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MATHEMATICAL QUESTIONS,

WITH THEIR

SOLUTIONS.

FROM THE "EDUCATIONAL TIMES."

VOL. XXVIII.

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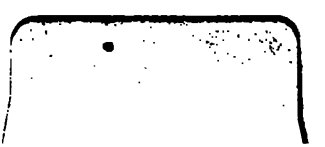

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Papers and Solutions not published in the "Educational Times."

EDITED BY

W. J. C. MILLER, B.A.,

REGISTRAR OF THE GENERAL MEDICAL COUNCIL.

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CONTENTS.

Mathematical Papers, &c.

No.		Page
123.	Symbolical or Abbreviated Language, with an Application to Mathematical Probability:—No. 1. By Hugh McColl, B.A...	20
124.	A New Method of Solving Equations of the Form of $x^2 - Ay^2 = 1$. By Dr. Hart	29
125.	On the Arithmetic of Compound Musical Tones. By Alexander J. Ellis, F.R.S.	34
126.	Note on Professor Tanner's Solution of Question 5314. By Alexander J. Ellis, F.R.S.	59
127.	Note on the Solution of Question 5314. By Professor Lloyd Tanner, M.A.	59
128.	Symbolical Language:—No. 2. By Hugh McColl, B.A. ...	100
129.	Letter, with Problem. By Professor Sylvester, F.R.S.....	106
130.	Note on Question 5461. By W. S. B. Woolhouse, F.R.A.S ...	109

Solved Questions.

2031.	(The Editor.)—Find the least positive value of x that will make the expression $927x^2 - 1236x + 413$ a rational square.....	65
3156.	(Professor Evans, M.A.)—Find a positive integral value of x that will satisfy the condition $94x^2 + 57x + 34 = \square$.	101
3192.	(J. M. Greenwood.) — The axes of two unequal cylinders intersect at right angles; determine the surface and volume common to both.....	68
3905.	(Professor Wolstenholme, M.A.)—If A, B, C be three points on a circle, there are generally four points P, P′, Q, Q′ on the circle such that $PA^2 = PB \cdot PC$, &c., the straight lines PP′, QQ′ being parallel to the external and internal bisectors of the angle A, and meeting the straight line BC in the same point as the tangent at A. The points Q, Q′ will coincide when $a^2 = 4bc$; and in this case the locus of Q (B, C being fixed points) is a rectangular hyperbola of which B, C are the foci, and the loci of P, P′ are two circular cubics whose foci are B, C and one vertex of the rectangular hyperbola. These cubics are inverses of the rectangular hyperbola with respect to its	

No. Page

vertices. The points Q, Q′ are impossible if $a^2 < 4bc$; but if $a^2 = 4n \cdot bc$, where $n > 1$, the loci of P, P′, Q, Q′ are all circular cubics having two foci at B, C. 71

4200. (The Editor.)—A variable square moves with its centre on the circumference of a given circle, and has one of its diagonals always parallel to a given diameter of the circle, and equal to twice the distance of the centre of the square from one end of the given diameter; show that the corners of the square move on the curves $r = 2a(\cos\theta \pm \cos 3\theta)$.....(1, 2), the given diameter ($=2a$) being the polar axis, and its end the origin. Trace these curves, showing that (1), with the upper sign, is the locus of the corners at the ends of the diagonal parallel to the given diameter; that the Cartesian equation of the curve is

$$(x^2+y^2)^2 = 4ax(x^2-y^2), \text{ or } \{y^2+x(2a+x)\}^2 = 4ax^2(a+2x)\ldots(3);$$

that the origin is a triple point where two branches form a crunode, cutting the axis at angles of 45°, and that the third branch, near the origin, takes the form of the parabola $y^2 = -4ax$; that the curve consists of three loops, one, of length $4a$ and breadth $2a$, to the right of the origin, and two equal loops extending to a distance $\frac{1}{2}a$ to the left of the origin:—also that (2), with the lower sign, is the locus of the ends of the diagonal perpendicular to the fixed diameter; that the Cartesian equation of the curve is

$$(x^2+y^2)^2 = 8axy^2, \text{ or } \{y^2-2x(4a-x)\}^2 = 8ax^2(2a-x)\ldots(4);$$

that the origin is a triple point where two of the branches form a ceratoid cusp, which, near the origin, takes the form of the semi-cubical parabola $x^3 = 8ay^2$, and the third branch, near the origin, takes the form of the parabola $y^2 = 8ax$; that the curve consists of two equal loops which extend to a distance of $2a$ to the right of the origin, and cut the large-loop branches of (1) on the circle, at an angular ordinate of 30°; that the chord of (1) parallel to the given diameter, and the chord of (2) perpendicular to the given diameter, are each double the corresponding chord of the circle; and that the whole area of each curve is double the area of the fixed circle. 60

4224. (Dr. Hart.)—Find three numbers such that, if the product of every two be added to the sum of the same two, the sums shall be squares; and also, if the product of every two be added to the third, the sums shall be squares. 67

4229. (The Editor.)—Find rational plane triangles, whose sides are integers, such that two of the sides shall differ by unity; and show that 9, 10, 17; 13, 14, 15; 11, 13, 20; 17, 25, 26 are sides of four such triangles..... 66

4561. (G. Sanderson.)—A great circle of a sphere is completely surrounded by 10 balls each 2 inches in diameter; and another row of 10 equal balls lies on the sphere, each one touching two of the first row and also touching each other; find the diameter of the second-row balls. 98

No. Page

4743. (S. Tebay, B.A.)—Two persons A and B, whose speeds of walking can vary from 0 to a and from 0 to b miles an hour respectively, start from two points A, B at the same time, and walk uniformly in a straight line for a time $= \frac{c}{a-b}$, where $c = AB$. Show that the probability that A overtakes B is

$$\frac{c}{a}\left\{\frac{b}{2a}\cdot\frac{2a-b}{a-b}-\log\frac{a}{a-b}\right\} \quad \ldots\ldots 23$$

4830. (T. Cotterill, M.A.)—Let a function of any letters be called alternating which changes its sign only, when any two of the letters are transposed, and be denoted by the letters being placed within parentheses. If the function is of such a nature that the equation

$$(pxy)(pzt)+(pyz)(pxt)+(pzx)(pyt)=0$$

holds for any values of letters, prove that the expression

$$\left\{(acd)(aef)(bde)(bcf)-(bcd)(bef)(ade)(acf)\right\}$$

is alternating, and including itself can be put under fifteen different forms. 26

4836. (Professor Wolstenholme, M.A.)—Three conics S_1, S_2, S_3 are such that the polar reciprocal of any one with respect to another is the third; a triangle ABC is inscribed in S_1 and its sides touch S_2 in a, b, c; prove that the triangle abc is self-conjugate to S_1, and that its sides touch S_3; also the tangents to S_1 at A, B, C form a triangle A′B′C′ which is self-conjugate to S_2 and whose angular points lie on S_3; also the tangent at A′ passes through b, c, &c. 97

4915. (Dr. Collins.)—Prove that (1) the equation of the parabolic asymptote of

$$ay^{n-2}x^2+a'y^{n-3}x^3+a''y^{n-4}x^4+\&c.+by^{n-1}+b'y^{n-2}x+b''y^{n-3}x^2+\&c.$$
$$+cy^{n-2}+c'y^{n-3}x+\&c.=0$$

is $$ax^2+by+\frac{ab'-a'b}{a}x+c+\frac{b^2(aa''-a'^2)+ab(a'b-ab'')}{a^3}=0;$$

(2) the parabolic asymptote of third order of

$$ay^{n-3}x^3+a'y^{n-4}x^4+a''y^{n-5}x^5+\&c.+by^{n-1}+b'y^{n-2}x+b'y^{n-3}x^2+\&c.$$
$$+cy^{n-2}+c'y^{n-3}x+c''y^{n-4}x^2+\&c.+dy^{n-3}+d'y^{n-4}x+\&c.=0$$

is $$a^3x^3+a^2by^2+a(ab-a'b)xy$$
$$+\left\{a(ab''-a''b)-a'(ab'-a'b)\right\}x^2+Ay+Bx+C=0;$$

(3) in the former case, when c is not $= 0$, the asymptotic common parabola above indicated has really a five-pointic contact at infinity with the given curve; and (4) find A, B, C in the second case (which is not done by Euler in pp. 98 and 102 of his *Analysis Infinitorum*) so that the cubic asymptote partly indicated above may really have a nine-point contact at infinity with the given curve of the nth order when b is not $= 0$; for when $b = 0$ in the first equation, the line projected to infinity is not a tangent to the curve, but a straight line passing through a double point of it; and when $b = 0$ in the second equation,

No. Page

then a tangent at a double point (and not at a point of inflection) has been projected to infinity..... 70

5013. (Professor Townsend, F.R.S.)—The several particles m, m', &c., of a material system moving freely in space, being supposed to come in simultaneous collision each with every other particle of the system, shew that the entire loss of *vis viva* consequent on their collision

$$= \frac{1-e^2}{\Sigma(m)} \cdot \Sigma\left\{nm'\left[(u-u)^2+(v-v')^2+(w-w')^2\right]\right\};$$

where u, v, w, u', &c. are the components of the velocities of m, m', &c. before collision, and e the coefficient of elasticity supposed the same for all the particles of the system. 94

5076. (Rev. A. F. Torry, M.A.)—A chord PQ cuts off a constant area from a given oval curve; show that the radius of curvature of its envelop will be $\frac{1}{4}$PQ $(\cot\theta+\cot\phi)$, where θ and ϕ are the angles at which PQ cuts the given curve. 28

5092. (R. Battle, M.A.) — If a given spherical quadrilateral be divided by an arc of a great circle into two quadrilaterals, prove independently of the principles of gnomonical projections that the intersections of the diagonals of these two are on the circumference of a great circle which passes through a given point..... 102

5118. (R. F. Scott, B.A)—If any binary quantic be transformed by means of a linear substitution, prove that any invariant of the transformed quantic is equal to its value for the original quantic multiplied by a power of the determinant of the substitution equal to the weight of the invariant 40

5140 (R. W. Genese, M.A.)—Construct geometrically the chords which cut an ellipse at given angles α, β; and shew there are in general eight solutions 63

5144. (J. C. Malet, M.A.)—If we denote by $\mathrm{F}(m,n,p)$ the determinant

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha^m & \beta^m & \gamma^m & \delta^m \\ \alpha^n & \beta^n & \gamma^n & \delta^n \\ \alpha^p & \beta^p & \gamma^p & \delta^p \end{vmatrix}$$

prove the formula $(\alpha+\beta+\gamma+\delta)\,\mathrm{F}(m,n,p)$
$= \alpha\beta\gamma\delta\,\mathrm{F}(m-1,n-1,p-1)+\mathrm{F}(m+1,n,p)+\mathrm{F}(m,n+1,p)$
$+\mathrm{F}(m,n,p+1)$;
and generalise for a similar determinant of any order..... 51

5148. (Rev. J. Blissard, B.A.)—Prove that

$$1^6-2^6+3^6-\ldots\pm x^6 = \tfrac{1}{2}(-1)^{x+1}(x^6+3x^5-5x^3+3x) \ldots\ldots(1);$$

$$(1.2.3)^2-(2.3.4)^2+\ldots \pm\left\{x(x+1)(x+2)\right\}^2 =$$

$$\tfrac{1}{2}(-1)^{x-1}(x^6+9x^5+28x^4+33x^3+7x^2-6x). \ldots\ldots(2).$$ 27

5163. (A. W. Panton, M.A.)—If the coordinates (x,y,z) of a point on a unicursal cubic are proportional to $a_1\theta^3+b_1\theta^2+c_1\theta+d_1$,

No. Page

$a_2\theta^3 + b_2\theta^2 + c_2\theta + d_2$, $a_3\theta^3 + b_3\theta^2 + c_3\theta + d_3$; show that the values of θ at the points of inflexion are the roots of the equation

$$\begin{vmatrix} 1 & -3\theta & 3\theta^2 & -\theta^3 \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix} = 0 \ldots\ldots 43$$

5164. (J. J. Walker, M.A.)—If

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2lx + 2my + 2nz + d = 0$$

be the equation to a pair of planes, prove that

1. The direction-cosines of the angles α, β, γ which the line of intersection of the planes makes with the (rectangular) axes are given by $\sec\alpha : \sec\beta : \sec\gamma = gh - af : hf - bg : fg - ch$.

2. The product of the perpendiculars let fall from the point (x', y', z') on the planes is equal to

$$\frac{ax'^2 + \ldots + 2fy'z' + \ldots + 2lx' + \ldots + d}{\left\{(a+b+c)^2 + 4(f^2-bc) + 4(g^2-ca) + 4(h^2-ab)\right\}^{\frac{1}{2}}} \ldots\ldots 56$$

5165. (R. A. Roberts, M.A.)—Prove that the envelop of the director circles of a series of conics passing through four fixed points is a bicircular quartic of which the vertices of the fixed triangle that is self-conjugate with regard to the conics are foci. 81

5169. (J. L. McKenzie.)—Two parabolas have a common point O, and the diameter of each through O is a tangent to the other; a line drawn through O meets one parabola in Q and the other in R; and a point P is taken on the line so that $OP^2 = OQ \cdot OR$. Prove that the locus of P is an hyperbola having the diameters of the parabolas through O as asymptotes. 85

5175. (Rev. H. G. Day, M.A.)—From a random point within the area of an acute-angled triangle, perpendiculars are drawn on the sides; show that the average area of the triangle formed by joining the feet of these perpendiculars is, in parts of the area of the given triangle, $\frac{1}{18}(\sin^2 A + \sin^2 B + \sin^2 C)$.... 82

5190. (By J. J. Walker, M.A.)—Referring to Professor Townsend's Quest. 5133, if ρ is the ratio of the density of the cone to that of the fluid in a case in which it can float in free equilibrium, with its axis inclined to the vertical at an angle ϕ, and if θ is the semi-angle of the cone, prove (1) that $\cos^2\phi = \frac{\sin^2\theta}{1 - \rho^{\frac{1}{3}}\sec^2\theta}$, and (2) that the base will be clear of the water when

$$\rho = (\cos 2\theta)^{\frac{3}{2}} \ldots\ldots 77$$

5203. (R. Tucker, M.A.)—Three particles are projected in the same vertical plane with the same velocity, but at different elevations; prove that the triangle formed by joining the foci is double that formed by joining the vertices of the parabolic paths..... 29

5204. (H. M. Dyer, B.A.)—If a semi-cubical parabola and a parabola have a common tangent at the vertices, and parallels to this tangent cut the semi-cubical in A', B', C', and the

No. Page

parabola in A, B, C respectively; show that the ratio $\Delta A'B'C' : \Delta ABC$ varies as the sum of the products taken two at a time of the ordinates of A, B, C..... 83

5211. (Professor Townsend, F.R.S.)—A circular cylinder, whose centre of inertia is not on its axis of figure, being supposed to roll without sliding on a rough horizontal plane, and to have the velocity of rotation just sufficient to carry it with exhausted energy to its position of unstable equilibrium against the action of gravity; determine, in finite terms, its time of passage from the opposite position of stable equilibrium to any other on its way..... 88

5223. (J. J. Walker, M.A.)—Show that, for all systems of rectangular axes drawn through a given point in a given plane area, in its plane, $\left\{\iint (x^2-y^2)\,dx\,dy\right\}^2 + 4\left\{\iint xy\,dx\,dy\right\}^2$, taken over the whole of the area, is constant; and that for a triangle, the point being its centroid, this constant value is $(\frac{1}{18}\Delta)^2 (a^4+b^4+c^4-b^2c^2-c^2a^2-a^2b^2)$..... 53

5226. (S. Tebay, B.A.)—A person has a rectangular plate of metal, length a and breadth b inches, valued at m shillings a square inch. Equal squares are cut from the corners, and the ends and sides turned up to form a vessel, which is filled with a material at n shillings a cubic inch. If the vessel be formed at random, find the average value of the box and contents; if with the greatest skill, find the depth of the vessel. 25

5247. (Professor Lloyd Tanner, M.A.)—If the determinant $\begin{vmatrix} a_1, & a_2, & \dots & a_n \\ b_1, & b_2, & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}$ be represented by $|\ |$; and $\begin{vmatrix} \iota, & \lambda, & \dots \\ j, & l, & \dots \end{vmatrix}$ represent the result of replacing the j^{th}, l^{th}, ... columns of $|\ |$ by $a_\iota, \beta_\iota, \dots, a_\lambda, \beta_\lambda, \dots, \dots$; prove that

$$\begin{vmatrix} \begin{vmatrix} \iota \\ j \end{vmatrix}, & \begin{vmatrix} \iota \\ l \end{vmatrix}, & \dots\dots \\ \begin{vmatrix} \lambda \\ j \end{vmatrix}, & \begin{vmatrix} \lambda \\ l \end{vmatrix}, & \dots\dots \\ \dots & \dots & \dots\dots \\ \dots & \dots & \dots\dots \end{vmatrix} = \left\{|\ |\right\}^{-1} \times \begin{vmatrix} \iota, & \lambda, & \dots \\ j, & l, & \dots \end{vmatrix},$$

the determinant on the left being of the r^{th} order..... 41

5249. (Professor Hall, M.A.)—Find the approximate value of the integral $\int_0^{\frac{1}{2}\pi} (\sin\theta)^{\frac{1}{2}}\,d\theta$ 19

5255. (C. W. Merrifield, F.R.S.)—Show that

$$a_n (a_0, a_1, \dots a_{2n} \mathcal{X} x, y)^{2n}$$

$$\equiv \left\{a_0 x^n - a_n (-y)^n\right\}\left\{a_n x^n - a_{2n} (-y)^n\right\}$$

$$+ 2nxy\left\{a_1 x^{n-1} - a_n (-y)^{n-1}\right\}\left\{a_n x^{n-1} - a_{2n-1} (-y)^{n-1}\right\} + \dots + Qx^n (-y^n),$$

where Q is the quadrinvariant..... 75

No. Page

5260. (W. H. H. Hudson, M.A.)—Prove that the radius of curvature of $au = \phi(\theta)$, at the point $\theta = a$, subtends at the pole the angle

$$\tan^{-1}\left\{\frac{\phi'(a)}{\phi(a)} \cdot \frac{\phi(a)^2 + \phi'(a)^2}{\phi'(a)^2 - \phi(a)\,\phi''(a)}\right\}. \quad \ldots\ldots\ldots\ldots \quad 39$$

5262. (S. Roberts, M.A.)—If s_0 is the number of integers less than N and prime to it, and s_1, s_2, s_3 are sums of those integers, of their squares and cubes respectively, prove that the roots of

$$s_0 a^3 + 3s_1 a^2 + 3s_2 a + s_3 = 0$$

are in arithmetical progression..... 45

5273. (Professor Crofton, F.R.S.)—If a point is taken at random in a triangle ABC, and three other points are taken at random; show that (1) the probability that all three fall on one of the triangles BXC, CXA, AXB is $\frac{3}{10}$; and (2) the probability that they fall, one on each triangle, is $\frac{1}{10}$. 55

5277. (Professor Lloyd Tanner, M.A.)—A number consists of n digits; its square ends with the same n digits in the same order; show that (1) for any given value of n there are two and only two such numbers in each of the scales of notation whose radices are 6, 10, 12 respectively; and (2) in any other scale (radix not greater than 12) no number (but 1 or 0) exists having this property. 32

5281. (W. S. B. Woolhouse, F.R.A.S.)—Logarithmic tables always give the nearest figure, so that the error in the last digit is always comprised within the limits $\pm\frac{1}{2}$. When any number (n) of logarithms are taken promiscuously and added together, determine the probability that the error of the sum will be contained within the limits $\pm k$, k being any given value less than $\frac{1}{2}n$. 78

5288. (Rev. A. F. Torry, M.A.)—Taking the eccentricity of the earth's orbit round the sun to be $\frac{1}{60}$, and the year to consist of $365\frac{1}{4}$ days; determine the greatest and least angular velocities of the earth about the sun..... 80

5293. (R. E. Riley, B.A.)—Chords of constant length ($2c$) are drawn in an ellipse; prove that the locus of their poles with regard to the ellipse is the curve

$$b^4(b^2-c^2)\,x^4 + a^4(a^2-c^2)\,y^4 + a^2b^2(a^2+b^2-2c^2)\,x^2y^2 = a^2b^2(a^4y^2 + b^4x^2). \quad 96$$

5294. (J. L. McKenzie, B.A.)– Through a fixed point A on a circle three fixed lines are drawn, meeting the circle in B_1 B_2 B_3. A line is drawn through B_1, parallel to a fixed direction, and meeting the circle on C_1; and through C_1 a line S_1 is drawn parallel to B_2B_3. Prove that the three lines S obtained by permuting B_1 B_2 B_3 intersect in a point on the circle. 28

5295. (J. F. Moulton, M.A.)—An indefinitely thin shell in the form of an infinite elliptical cylinder, the section of the two surfaces being similar, similarly situated, and concentric curves, is surrounded by homogeneous fluid which it attracts. Show that the pressure at any point is given by $2m\rho \log \frac{a+b}{a'+b'}$, where a', b', a, b are the semi-axes of the confocal cylinders through the point and bounding the fluid respectively. 79

No. Page

5303. (Sir James Cockle, F.R.S.)—Given

$$\frac{d^2z}{dx^2} - a^2y^{2b}\frac{d^2z}{dy^2} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1);$$

find formulæ of reduction..... 47

5306. (Professor Cayley, F.R.S.)—If α, β, γ, δ; α_1, β_1, γ_1, δ_1 are such that $(\alpha_1-\delta_1)(\beta_1-\gamma_1) = (\alpha-\delta)(\beta-\gamma)$,
$(\beta_1-\delta_1)(\gamma_1-\alpha_1) = (\beta-\delta)(\gamma-\alpha)$, $(\gamma_1-\delta_1)(\alpha_1-\beta_1) = (\gamma-\delta)(\alpha-\beta)$; show that the three equations

$$\frac{x_1-\alpha_1}{x_1-\delta_1} = \frac{1}{(\beta_1-\delta_1)(\gamma_1-\delta_1)}\left\{(x-\alpha)^{\frac{1}{2}}(x-\delta)^{\frac{1}{2}} - (x-\beta)^{\frac{1}{2}}(x-\gamma)^{\frac{1}{2}}\right\}^2,$$

$$\frac{x_1-\beta_1}{x_1-\delta_1} = \frac{1}{(\gamma_1-\delta_1)(\alpha_1-\delta_1)}\left\{(x-\beta)^{\frac{1}{2}}(x-\delta)^{\frac{1}{2}} - (x-\gamma)^{\frac{1}{2}}(x-\alpha)^{\frac{1}{2}}\right\}^2,$$

$$\frac{x_1-\gamma_1}{x_1-\delta_1} = \frac{1}{(\alpha_1-\delta_1)(\beta_1-\delta_1)}\left\{(x-\gamma)^{\frac{1}{2}}(x-\delta)^{\frac{1}{2}} - (x-\alpha)^{\frac{1}{2}}(x-\beta)^{\frac{1}{2}}\right\}^2,$$

are equivalent to each other; and show also that, consistently with the foregoing relations between the constants, the differences $\alpha_1-\delta_1$, $\beta_1-\delta_1$, $\gamma_1-\delta_1$ may be so determined that the equations in (x, x_1) constitute a particular integral of the differential equation

$$\frac{dx}{\left\{(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)\right\}^{\frac{1}{2}}} = \frac{dx_1}{\left\{(x_1-\alpha_1)(x_1-\beta_1)(x_1-\gamma_1)(x_1-\delta_1)\right\}^{\frac{1}{2}}}.$$

..... 17

5307. (Professor Crofton, F.R.S.)—In any plane figure which has a centre, O, if any diameter AOB be drawn, and G be the centroid of either half into which it divides the area; then, if P be the perimeter of the locus of G, prove that the mean distance of the centre O from all points in the area is equal to $\frac{1}{4}$P..... 20

5308. (Professor Townsend, F.R.S)—A slender circular cord, of uniform thickness and arbitrary length, being supposed to intersect at right angles, in a plane perpendicular to its axis, a thin circular tube, of arbitrary radius and infinite length, repelling it according to the law of the inverse fourth power of the distance; show that the cord will, without deformation at any point, be held in free equilibrium by the repulsion of the tube. 30

5311. (The Editor.)—If, in a triangle ABC, P be the orthocentre, O_1, O_2, O_3 the centres of the escribed circles, and Q the centre of the circumscribed circle; prove that

$$\begin{vmatrix} 1, & 1, & 1 \\ \overline{PO_1}^2, & \overline{PO_2}^2, & \overline{PO_3}^2 \\ \overline{QO_1}^2, & \overline{QO_2}^2, & \overline{QO_3}^2 \end{vmatrix} = \frac{a^2b^2c^2}{r^2}\left(\frac{b-c}{a} + \frac{c-a}{b} + \frac{a-b}{c}\right) \ldots 95$$

5312. (The Editor.)—If a, b, c be the sides of a triangle; R, r, r_1, r_2, r_3 the radii of the circumscribed, inscribed, and escribed circles; and ρ the radius of the circle inscribed in the orthocentric triangle; prove that

$$4(\mathrm{R}^2 + 2\mathrm{R}r_1) - (a^2 + b^2 + c^2 - bc + ca + ab) = 2(r_1^2 - \mathrm{R}\rho),$$

with two analogous formulæ in r_2 and r_3. 24

No. Page

5313. (Elizabeth Blackwood.)—If A, B, C, D are random points in the perimeter of a square ; find the chance that the straight lines AB, CD will intersect within the square. 85

5314. (Alexander J. Ellis, F.R.S.)—Given $f(ax, y) = af(x, y)$ and $f(x, by) = bf(x, y)$, to find all the possible forms of $f(x, y)$. It is usually assumed that $f(x, y) = cxy$, where c is a constant, and this form evidently satisfies the conditions. But if $a = b$, then $f(x, y) = c \div xy$ also satisfies them; and generally, if $a^m = a$, $b^n = b$, that is, if a is any one of the roots of a, and b any one of the roots of b, then $f(x, y) = x^m y^n$ also satisfies the conditions. The question becomes important in such problems as this:—The work done in carrying a load over a distance is multiplied when either the load or the distance is multiplied; what is the relation between work done, load, and distance? 58, 83

5317. (S. Roberts, M.A.)—Prove that the locus of the poles of the tangents of a cubic curve $U = 0$ is

$$TU^3 - 20U^2H + 28H^3 = 0,$$

where H is the Hessian and S, F are the usual invariants..... 77

5319. (C. Taylor, M.A.)—In the right circular cone, if C, C′ be the centres of a pair of focal spheres, show that the sphere on CC′ contains the auxiliary circle of the section 25

5325. (S. Tebay, B.A.)—Show that the whole attractive force on an undisturbed planet varies as the angle swept out by the radius vector; and thence show that there can be no accumulated resultant force in any direction when the planet has performed a complete revolution. 90

5327. (Professor Sylvester, F.R.S.)—If P, Q, R are three points in a linkwork attached by rigid rods to a fourth point O in the same plane, construct a system of jointed rods connected with the given one, such that in it a certain point G shall always remain the centroid of the triangle PQR. 24

5340. (C. Leudesdorf, M.A.)—Find an expression for s_n, the sum of the n^{th} powers of the numbers less than a given number N, and prime to it; and hence prove that, when n is odd, the equation $(s_0 s_1 s_2 \ldots s_n \overline{\square} x, 1)^n = 0$ is satisfied by $x = -\frac{s_1}{s_0}$, and that the other roots are all of the form $-\frac{s_1}{s_0} \pm y$..... 103

5342. (W. Gallatly, M.A.)—If S be the focus of a conic inscribed in a triangle ABC, and any tangent meet the perpendiculars to SA, SB, SC in the points P, Q, R respectively, prove that AP, BQ, CR meet in a point. 99

5352. (W. C. Otter, F.R.A.S.)—A gentleman has a circular meadow whose area is just five acres, surrounded by an iron palisading, to the inside of which he wishes to tether his horse so as to enable him to graze over just *one* acre of ground: find the length of the tether. 101

No. Page

5356. (A. Winslow.)—If any three circles touch each other, show that the three lines joining the centre of each circle to the point of tangency of the other two intersect in the same point. 80

5359. (Professor Townsend, F.R.S.)—Two ordinary tangents being supposed drawn to a cuspidal cubic from any point on the cuspidal tangent, or to a tricuspidal quartic from any point on one of the three cuspidal tangents; shew that their chord of contact passes through the point of inflexion in the case of the cubic, and through the pole of the cuspidal tangent with respect to the bitangent chord in the case of the quartic. 112

5363. (Professor Lloyd Tanner, M.A.)—Solve the equation

$$\frac{d^2z}{dxdy} - \frac{1}{y}\frac{dz}{dx} + ue^{-z}\frac{dz}{dy} + \left(\frac{u}{y} - \frac{du}{dy}\right)e^{-z} = 0;$$

where u is any function of x, y..... 63

5386. (S. A. Renshaw.)—Let ABCD be a quadrilateral inscribed in a conic of focus F, and BCD any one of the four triangles of which it is composed; and let BC, CD, DB be produced to meet the directrix in P_2, P_3, P_6; and FP_2, FP_3, FP_6 be joined, and FQ_2, FQ_3, FQ_6 be drawn perpendicular to them, meeting the directrix in Q_2, Q_3, Q_6; then prove that (1), if Q_2A, Q_3A, Q_6A be joined meeting BC, CD, and DB, in a_1, a_2, a_3, these three points a_1, a_2, a_3 will be in a straight line; (2) if each of the remaining triangles CDA, DAB, CDB be similarly treated in conjunction with the vertices B, C, D (*i. e.*, CDA with B, &c.), there will be obtained similarly three other sets of three points each, viz., (b_1, b_2, b_3), (c_1, c_2, c_3), (d_1, d_2, d_3), each set lying in a straight line, and the four lines containing the four sets will all pass through the same point O. 57

5394. (Professor Minchin, M.A.)—If a curve and its inverse be described with the same law of force, viz., $\mu r^5 p^{-5}$; prove that both curves are included in the class whose equation is $r^6 + ap^3 + b = 0$. 76

5400. (J. W. L. Glaisher, F.R.S.)—Verify that

$$u = \int_{a^{\frac{1}{n}}x^{\frac{1}{n}}}^{\infty} e^{-z^n - \frac{a^n x^n}{z^n}} dz$$

is a particular integral of the differential equation

$$\frac{d^2u}{dx^2} - n^2 a^n x^{n-2} u = \tfrac{1}{4} e^{-2a^{\frac{1}{2}n} x^{\frac{1}{2}n}} . a^{\frac{1}{2}} x^{-\frac{1}{2}},$$

which is a particular case of the extended R ccati's equation

$$\frac{d^2u}{dx^2} - x^q u = f(x) ..... 76$$

5409. (E. B. Seitz.)—A semicircle is divided into two quadrants, and a point is taken at random in each quadrant; find the chance that their distance apart is less than the radius of the semicircle..... 110

5412. (R. W. Genese, M.A.)—Prove that the product of the two normals that can be drawn to a parabola from a point on it is equal to 2SP . PG. 93

No. Page

5413. (Rev. A. F. Torry, M.A.) — Two particles describe the same ellipse subject to the same force in the centre: show that their directions of motion at any time intersect on a similar ellipse. 95

5429. (Professor Lloyd Tanner, M.A.)—If

$$\frac{d^2z}{dx^2}\cdot\frac{d^2z}{dy^2}-\left(\frac{d^2z}{dx\,dy}\right)^2=0;$$

then, x being considered as a function of z, y,

$$\frac{d^2x}{dz^2}\cdot\frac{d^2x}{dy^2}-\left(\frac{d^2x}{dz\,dy}\right)^2=0. \quad \text{..... } 90$$

5438. (The Editor.) — If A, B, C, D be taken at random on the perimeter of a regular n-gon, find the respective probabilities that AB, CD will intersect (1) inside, (2) on, (3) outside the perimeter; also (4) what these values become when n is increased without limit; and (5) deduce therefrom solutions to Questions 5313, 5439, 5461. 106

5439. (Elizabeth Blackwood.)—Four random points A, B, C, D are taken in the perimeter of an equilateral triangle; show that the chance that the straight lines AB, CD will intersect *inside* the triangle (intersections on the perimeter *not* being counted within) is $\frac{8}{9}$. 99

5461. (The Editor.) — All possible chords of a given circle being supposed equally probable; required the probabilities (1) of the intersection of two of them taken at random being within the circle, (2) of the intersections of three of them taken at random being all within the circle. 108

5482. (S. Constable.)—Let ABC be a triangle (of area Δ) circumscribed by a circle, and let the arcs BC, CA, AB be bisected at A′, B′, C′, forming a new triangle A′B′C′ ($=\Delta_1$), and a hexagon AC′BA′CB′ ($=\Delta_2$); also let Δ_3 denote the area of the triangle $O_1O_2O_3$, where O_1, O_2, O_3 are the escribed centres; then prove that

$$\Delta : \Delta_1 : \Delta_2 : \Delta_3 = r : \tfrac{1}{4}R : R : 2R. \quad \text{..... } 112$$

CORRIGENDA.

VOL. XXVII.

Page 19, line 3 from bottom, *for* yu_1 *read* $\frac{u_1}{y}$.

Page 85, line 2 from bottom, *for* RP *read* $R\rho$.

Page 87, line 16, *for* $2R^2$ *read* $2r^2$.

Page 109, line 14, *dele* [.

VOL. XXVIII.

Page 25, line 5, *for* $s-a$ *read* $s-a$.

MATHEMATICS

FROM

THE EDUCATIONAL TIMES.

WITH ADDITIONAL PAPERS AND SOLUTIONS.

5306. (By Professor CAYLEY, F.R.S.)—If $\alpha, \beta, \gamma, \delta$; $\alpha_1, \beta_1, \gamma_1, \delta_1$ are such that

$$(\alpha_1-\delta_1)(\beta_1-\gamma_1) = (\alpha-\delta)(\beta-\gamma),$$

$$(\beta_1-\delta_1)(\gamma_1-\alpha_1) = (\beta-\delta)(\gamma-\alpha), \quad (\gamma_1-\delta_1)(\alpha_1-\beta_1) = (\gamma-\delta)(\alpha-\beta);$$

show that the three equations

$$\frac{x_1-\alpha_1}{x_1-\delta_1} = \frac{1}{(\beta_1-\delta_1)(\gamma_1-\delta_1)}\left\{(x-\alpha)^{\frac{1}{2}}(x-\delta)^{\frac{1}{2}}-(x-\beta)^{\frac{1}{2}}(x-\gamma)^{\frac{1}{2}}\right\}^2,$$

$$\frac{x_1-\beta_1}{x_1-\delta_1} = \frac{1}{(\gamma_1-\delta_1)(\alpha_1-\delta_1)}\left\{(x-\beta)^{\frac{1}{2}}(x-\delta)^{\frac{1}{2}}-(x-\gamma)^{\frac{1}{2}}(x-\alpha)^{\frac{1}{2}}\right\}^2,$$

$$\frac{x_1-\gamma_1}{x_1-\delta_1} = \frac{1}{(\alpha_1-\delta_1)(\beta_1-\delta_1)}\left\{(x-\gamma)^{\frac{1}{2}}(x-\delta)^{\frac{1}{2}}-(x-\alpha)^{\frac{1}{2}}(x-\beta)^{\frac{1}{2}}\right\}^2,$$

are equivalent to each other; and show also that, consistently with the foregoing relations between the constants, the differences $\alpha_1-\delta_1$, $\beta_1-\delta_1$, $\gamma_1-\delta_1$ may be so determined that the equations in (x, x_1) constitute a particular integral of the differential equation

$$\frac{dx}{\left\{(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)\right\}^{\frac{1}{2}}} = \frac{dx_1}{\left\{(x_1-\alpha_1)(x_1-\beta_1)(x_1-\gamma_1)(x_1-\delta_1)\right\}^{\frac{1}{2}}}.$$

Solution by ROBERT RAWSON.

Squaring the quantity on the right-hand side, and arranging the term on the left-hand side, we have the following three equations:—

$$(\beta_1-\delta_1)(\gamma_1-\delta_1) - \frac{(\alpha_1-\delta_1)(\beta_1-\delta_1)(\gamma_1-\delta_1)}{x_1-\delta_1}$$

$$= 2x^2-(\alpha+\beta+\gamma+\delta)x+\alpha\delta+\beta\gamma-2\left\{(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)\right\}^{\frac{1}{2}} \ldots (1),$$

$$(\gamma_1-\delta_1)(\alpha_1-\delta_1)-\frac{(\beta_1-\delta_1)(\gamma_1-\delta_1)(\alpha_1-\delta_1)}{x_1-\delta_1}$$

$$=2x^2-(\alpha+\beta+\gamma+\delta)x+\beta\delta+\gamma\alpha-2\left\{(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)\right\}^{\frac{1}{2}}\ldots(2),$$

$$(\alpha_1-\delta_1)(\beta_1-\delta_1)-\frac{(\gamma_1-\delta_1)(\alpha_1-\delta_1)(\beta_1-\delta_1)}{x_1-\delta_1}$$

$$=2x^2-(\alpha+\beta+\gamma+\delta)x+\gamma\delta+\alpha\beta-2\left\{(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)\right\}^{\frac{1}{2}}\ldots(3).$$

Now it is readily seen that equations (1), (2), (3) are equivalent, providing the three following equations be satisfied:—

$$(\beta_1-\delta_1)(\gamma_1-\delta_1)-\alpha\delta-\beta\gamma=(\gamma_1-\delta_1)(\alpha_1-\delta_1)-\beta\delta-\alpha\gamma$$
$$=(\alpha_1-\delta_1)(\beta_1-\delta_1)-\gamma\delta-\alpha\beta\ldots\ldots(4).$$

The system (4) is seen, at once, to be satisfied by means of the three conditional equations amongst the constants as given by the question. Hence the three equations are equivalent to each other.

Again, extracting the square root of each side of the three equations in the question, and multiplying the three resulting equations together, we have

$$\frac{\left\{(x_1-\alpha_1)(x_1-\beta_1)(x_1-\gamma_1)(x_1-\delta_1)\right\}^{\frac{1}{2}}}{\left\{(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)\right\}^{\frac{1}{2}}}=\frac{(x_1-\delta_1)^2}{(\alpha_1-\delta_1)(\beta_1-\delta_1)(\gamma_1-\delta_1)}$$
$$\times\left\{4x-(\alpha+\beta+\gamma+\delta)-\frac{\left\{\begin{array}{l}(x-\beta)(x-\gamma)(x-\delta)+(x-\alpha)(x-\gamma)(x-\delta)\\+(x-\alpha)(x-\beta)(x-\delta)+(x-\alpha)(x-\beta)(x-\gamma)\end{array}\right\}}{\left\{(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)\right\}^{\frac{1}{2}}}\right\}$$
$$\ldots\ldots\ldots(5).$$

Differentiating either of the three equations (1), (2), (3), we have

$$\frac{(\alpha_1-\delta_1)(\beta_1-\delta_1)(\gamma_1-\delta_1)}{(x_1-\delta_1)^2}dx_1=$$
$$\left\{4x-(\alpha+\beta+\gamma+\delta)-\frac{\left\{\begin{array}{l}(x-\beta)(x-\gamma)(x-\delta)+(x-\alpha)(x-\gamma)(x-\delta)\\+(x-\alpha)(x-\beta)(x-\delta)+(x-\alpha)(x-\beta)(x-\gamma)\end{array}\right\}}{\left\{(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)\right\}^{\frac{1}{2}}}\right\}dx$$
$$\ldots\ldots\ldots(6).$$

From equations (5) and (6), by division, we obtain

$$\frac{dx}{\left\{(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)\right\}^{\frac{1}{2}}}=\frac{dx_1}{\left\{(x_1-\alpha_1)(x_1-\beta_1)(x_1-\gamma_1)(x_1-\delta_1)\right\}^{\frac{1}{2}}}\ldots(7).$$

Either of the equations is therefore a solution of the differential equation (7), whatever may be the values of the differences $(\alpha_1-\delta_1)$, $(\beta_1-\delta_1)$, $(\gamma_1-\delta_1)$.

By adding each side of the three conditional equations together, each side of the resulting equation vanishes identically.

The three conditional equations are therefore not independent; one of them is a consequence of the remaining two. This circumstance makes this very beautiful theorem more valuable than it would be if the conditional equations were independent.

5249. (By Professor HALL, M.A.)—Find the approximate value of the integral $\int_0^{\frac{1}{2}\pi} (\sin\theta)^{\frac{1}{2}}\, d\theta$.

Solution by J. W. L. GLAISHER, F.R.S.

Transforming the integral by taking $\sin\theta = x^{\frac{1}{2}}$, we have

$$\int_0^{\frac{1}{2}\pi} (\sin\theta)^{\frac{1}{2}}\, d\theta = \tfrac{1}{2}\int_0^1 x^{-\frac{1}{4}}(1-x)^{-\frac{1}{2}}\, dx,$$

$$= \frac{1}{2}\,\frac{\Gamma(\frac{3}{4})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{5}{4})} = \frac{2\sqrt{\pi}}{3}\,\frac{\Gamma(1{\cdot}75)}{\Gamma(1{\cdot}25)}.$$

Now $\log\Gamma(x)$ was tabulated by Legendre (*Traité des Fonctions Elliptiques*, T. ii., pp. 490—499) from $x = 1$ to $x = 2$ at intervals of 0·001 to twelve places; so that, taking the values of $\log\Gamma(1{\cdot}250)$ and $\log\Gamma(1{\cdot}750)$ from this table, we have (u denoting the integral)

$$\log u = 0{\cdot}078\ 507\ 652\ 449,$$

whence, by ten-figure logarithms, we have

$$u = 1{\cdot}198\ 140\ 236.$$

The last place both in $\log u$ and u is not, necessarily, quite accurate. Of course u could be obtained from $\log u$ to two places more if required.

In the same way it is readily seen that the integral

$$\int_0^{\frac{1}{2}\pi} (\sin\theta)^{\frac{1}{n}}\, d\theta = \frac{n\sqrt{\pi}}{n+1}\cdot\frac{\Gamma\left(1+\frac{1}{2}+\frac{1}{2n}\right)}{\Gamma\left(1+\frac{1}{2n}\right)},$$

to which (when n is integral) Legendre's table is at once applicable; and, in general, as is well known,

$$\int_0^{\frac{1}{2}\pi} (\sin\theta)^a (\cos\theta)^b\, d\theta = \frac{1}{2}\,\frac{\Gamma(\frac{1}{2}+\frac{1}{2}a)\,\Gamma(\frac{1}{2}+\frac{1}{2}b)}{\Gamma(1+\frac{1}{2}a+\frac{1}{2}b)}.$$

[Mr. CARR and others substitute $x = \sin\theta$, and obtain

$$\int_0^{\frac{1}{2}\pi} (\sin\theta)^{\frac{1}{2}}\, d\theta = \int_0^1 x^{\frac{1}{2}}(1-x^2)^{-\frac{1}{2}}\, dx.$$

Expanding by the Binomial Theorem, and integrating the terms separately, they obtain for the value of the integral the approximating series

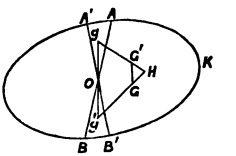
$$\frac{2}{3}+\frac{1}{7}+\frac{3}{44}+\frac{1}{24}+\frac{35}{1216}+\ldots\ldots,$$

the general term of which may be written

$$\frac{\lfloor 2n-1}{2^{2n-2}(4n+3)\,\lfloor\underline{n}\ \lfloor\underline{n-1}}.]$$

5307. (By Professor Crofton, F.R.S.)—In any plane figure which has a centre, O, if any diameter AOB be drawn, and G be the centroid of either half into which it divides the area; then, if P be the perimeter of the locus of G, prove that the mean distance of the centre O from all points in the area is equal to $\frac{1}{4}$P.

Solution by E. B. Elliott, M.A.



Take two consecutive positions AB, A'B' of AB, and let G, G' be the corresponding positions of G; also let g, g' be the centres of gravity of the small areas AOA', BOB'.

Then Og is the mean distance from O of points in the area AOA'.

Now G' is centre of gravity of the semi-area A'OB'K, and g that of its part, the small area AOA'; therefore that of its remainder, the area AOB'K, lies on gG' produced. Similarly, it lies on g'G produced; and therefore it is H, the intersection of gG' and g'G.

Therefore $$\frac{Hg}{HG'} = \frac{\text{Area A'OB'K}}{\text{Area AOA'}} = \frac{Hg'}{HG};$$

therefore GG' is parallel to gg'.

Therefore also $$\frac{GG'}{gg'} = \frac{\text{Area AOA'}}{\text{Area A'OB'K}};$$

or, denoting the entire area by Ω, and GG' by dP,

$$d\text{P} = \frac{4\,\text{AOA'}.\,Og}{\Omega}.$$

Therefore, summing, P = 4 times the mean of Og
= 4 times mean distance of points in Ω from O.

Symbolical or Abbreviated Language, with an Application to Mathematical Probability. *By* Hugh McColl, B.A.

Def. 1.—The symbols A, B, C, &c. denote *statements* (or propositions) registered for convenience of reference in a table. The equation A = 1 asserts that the statement A is *true;* the equation A = 0 asserts that the statement A is *false;* and the equation A = B asserts that A and B are equivalent statements.

Def. 2.—The symbol A × B × C or ABC denotes a *compound* statement, of which the statements A, B, C may be called the *factors*. The equation ABC = 1 asserts that *all three statements* are *true;* the equation ABC = 0 asserts that all three statements are *not* true, *i. e.*, that at least *one* of the three is false. Similarly a compound statement of any number of factors may be defined.

DEF. 3.—The symbol $A+B+C$ denotes an *indeterminate* statement, of which the statements A, B, C may be called the *terms*. The equation $A+B+C=0$ asserts that *all the three statements* are *false;* the equation $A+B+C=1$ asserts that all the three are *not* false, *i.e.*, that at least *one* of the three is true. Similarly $A+B$, $A+B+C+D$, &c. may be defined.

DEF. 4.—The symbol A' is the *denial* of the statement A. The two statements A and A' are so related that they satisfy the two equations $AA'=0$ and $A+A'=1$. The same symbol (*i.e.*, a *dash*) will convert any complex statement into its denial. For example, $(AB)'$ is the denial of the compound statement AB.

DEF. 5.—When an *indeterminate statement* is a factor of a compound statement, the former is enclosed in brackets. For example, $(A+B)(C+D)$ denotes a compound statement of two indeterminate factors. Exactly as in ordinary algebra, we have the identity

$$(A+B)(C+D)=AC+AD+BC+BD;$$

and the same rule may be extended to a compound statement of any number of indeterminate factors, and whatever be the number of terms in the respective factors.

The following examples will illustrate the preceding definitions:—

(1.) The equation $A=AB$ asserts either that the statement A implies the statement B, or else that B is true necessarily. As particular cases of this, we have $A=AA=AAA=$ &c., as repetition neither strengthens nor weakens the logical value of a statement. Again,

$$A=A(B+B')=A(B+B')(C+C')=\text{&c.};$$

for $B+B'=C+C'=1$ (def. 4).

(2). $$(AB)'=AB'+A'B+A'B'$$
$$=AB'+A'(B+B')=AB'+A', \text{ for } B+B'=1.$$

(3). $(AB)'=BA'+B'$, by exchanging A and B in (2).

(4). $(A+B)'=A'B'$; $(A+B+C)'=A'B'C'$, &c.

(5). $$\{AB(C+D)\}'=AB(C+D)'+(AB)', \text{ from (2)}$$
$$=ABC'D'+(AB)', \text{ from (4)}$$
$$=ABC'D'+AB'+A', \text{ from (2)}.$$

RULE 1.—If there be any statements (simple or complex), say A, B, C, ..., of such a nature that they neither strengthen nor weaken each other (the truth or falsehood of one in no way affecting the truth or falsehood of the rest); then the chance that the compound statement ABC ... is true is the *product abc* ..., in which a = the chance that A is true, b the chance that B is true, and so on.

RULE 2.—If there be any statements A, B, C, ... (simple or complex) of such a nature that only *one* of them can be true; then the chance that the *indeterminate statement* $A+B+C+\ldots$ is true is $a+b+c+\ldots$, in which a, b, c ... denote the same chances as in the preceding rule.

By an extension of the preceding principles and notation we may obtain a solution of the following general problem:—

Given that certain functions, P, Q, R, ... say, of any variables $x, y, z, \ldots$ are respectively positive (or negative), find the chance that certain other functions U, V, W, ... of the same variables are positive (or negative), the values of $x, y, z, \ldots$ being taken at random.

The method of effecting the solution may be briefly indicated as follows:—

DEF. 6.—The symbol p prefixed to any algebraical (or arithmetical) expression converts the expression into a statement, namely, that the expression is real and *positive;* the prefix p' in like manner asserts that the expression affected by it is real and *negative;* and the prefix q asserts that the expression affected by it is *impossible.* For example, $pxp'(y-x)\,q\{y-(m^2-x^2)^{\frac{1}{2}}\}$ is a compound statement of three factors: the first factor asserts that x is *positive;* the second asserts that $y-x$ is *negative;* and the third asserts that $y-(m^2-x^2)^{\frac{1}{2}}$ is *impossible.* Again, let it be given that x, y, a are all three real and positive, it is easy to perceive the identity

$$p(x^2+y^2-a^2) = p\{y-(a^2-x^2)^{\frac{1}{2}}\}\,p(a-x)+p'(a-x).$$

DEF. 7.—The symbols $x_1, x_2, x_3, \ldots$ denote the 1st, 2nd, 3rd, &c. limits of x, registered in any convenient order in a table of reference. The limits of the other variables of the expression (or expressions) under consideration are denoted similarly.

DEF. 8.—The symbol $x_{mnr's't'}$ is a compound statement of 5 factors, and asserts that x_m and x_n are *inferior* limits of x, while x_r, x_s, and x_t are *superior* limits of x. In fact, it is an abbreviation for the compound statement $p(x-x_m)\,p(x-x_n)\,p'(x-x_r)\,p'(x-x_s)\,p'(x-x_t)$. Compound statements of any number of factors, and having reference to the limits of x (or any other variable), may be similarly abbreviated.

RULE 3.—The compound statement $x_{m'n'r's'} = x_{m'}\mathrm{A}+x_{n'}\mathrm{B}+x_{r'}\mathrm{C}+x_{s'}\mathrm{D}$, in which $x_{m'}$, $x_{n'}$, &c. are abbreviations for the statements that the m^{th}, n^{th}, &c. limits of x are *superior* limits; while the capitals A, B, C, &c. respectively denote the statements that x_m is the nearest superior limit of x, that x_n is the nearest limit, that x_r is the nearest limit, and so on. In other words, A is an abbreviation for the compound statement $p'(x_m-x_n)\,p'(x_m-x_r)\,p'(x_m-x_s)$. The value of B is obtained from this by simply exchanging m and n; the value of C is obtained from it by exchanging m and r; and so on.

RULE 4.—This is obtained from the preceding rule by simply copying all the words in it (except *superior*, for which we must write *inferior*), and *omitting all the accents*, both on the numbers m, n, r, s and on the symbol p.

RULE 5.—The statement $x_{mn'} = x_{mn'}\mathrm{A}$, in which $\mathrm{A} = p(x_n-x_m)$.

By repeated applications of Rules 3, 4, 5, any compound statement of the form $x_{1.2'.3'.4.5.6'\ldots}\,y_{1'.2.3'.4\ldots}\,z_{1'.2'.3'.4.5\ldots}$, say, may be reduced either to a single elementary term of the form $x_{mn'}\cdot y_{rs'}\cdot z_{tu'}$, in which x_m and x_n are the *nearest limits* (inferior and superior) of x, and so on for y and z; or it may be reduced to an indeterminate statement consisting of several such terms. This effected, the integral calculus must do the rest. If all the values of x, y, and z between the assigned limits are equally probable, the chance that the elementary term given above is a true statement is proportional to the definite integral

$$\int_{x_m}^{x_n} dx \int_{y_r}^{y_s} dy \int_{z_t}^{z_u} dz.$$

In this notation it is evident that only one meaning is attached to the

symbol $x_{m'}$, which is always an abbreviation for the statement $p'(x-x_m)$, and asserts that x_m (the mth limit of x) is a *superior* limit. With the symbol x_m the case is different; in the table of limits, and when effected by the symbol p or p', it simply denotes the mth limit of x, as registered in the table; but in all other cases the symbol x_m denotes a *statement*, namely, the statement $p(x-x_m)$, and asserts that x_m (the mth limit of x) is an *inferior* limit. It thus appears that x_m sometimes denotes a *limit* and sometimes a *statement* with reference to that limit; but no ambiguity or confusion can result from this double meaning, as the context will always show what meaning is to be attached to the symbol. For example, if one side of an equation denotes a *statement*, the other side also must denote a statement; if one factor of an expression is a statement, all the factors must be statements; if one term of an expression is a statement, all the other terms of the expression must be statements. There is, therefore, no danger of confounding the *statement* x_m with the *limit* x_m, just as in ordinary language there is no fear of confounding the noun *bear* with the verb *bear* in such sentences as "the *bear* will *bear* his share of the burden."

The symbols px, $p'x$, py, $p'y$, &c., occur so frequently that they may be conveniently denoted always by x_0, $x_{0'}$, y_0, $y_{0'}$, &c.

In a compound statement, therefore, which contains x_0 as a factor, x is always positive; and in a compound statement which contains $x_{0'}$ as a factor, x is always negative. The same remarks of course apply to y_0, $y_{0'}$, z_0, $z_{0'}$, &c.

4743. (By S. Tebay, B.A.)—Two persons A and B, whose speeds of walking can vary from 0 to a, and from 0 to b miles an hour respectively, start from two points A, B at the same time, and walk uniformly in a straight line for a time $= \frac{c}{a-b}$, where $c = AB$. Show that the probability that A overtakes B is $\frac{c}{a}\left\{\frac{b}{2a}\cdot\frac{2a-b}{a-b} - \log\frac{a}{a-b}\right\}$.

Solution by Elizabeth Blackwood.

Let v be the speed of B, and t the time in which A overtakes B. The event will happen if the speed of A be not less than $v + ct^{-1}$; and

$$\int_0^b \left(1 - \frac{v+ct^{-1}}{a}\right)\frac{du}{b} = 1 - \frac{2c+bt}{at}.$$

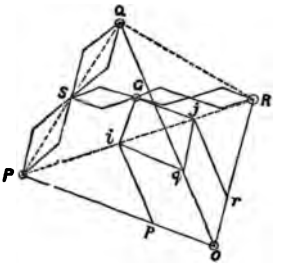
The limits of t are $\frac{c}{a}$ and $\frac{c}{a-b}$, and the integral gives the proposed expressions.

5327. (By Professor SYLVESTER, F.R.S.)—If P, Q, R are three points in a linkwork attached by rigid rods to a fourth point O in the same plane, construct a system of jointed rods connected with the given one, such that in it a certain point G shall always remain the centroid of the triangle PQR.

I. *Solution by* G. S. CARR.

Connect the points P, Q by a "lazy tongs" formed of two parallelograms PS, SQ; and connect the points S, R by a "lazy tongs" of three parallelograms, as shown in the figure. The point G will always be the centroid of the triangle PQR, whatever motion may be given to its angular points.

In the same way, by increasing the number of parallelograms, any commensurable divisions of the lines PQ, SR may be preserved throughout the motion of the system.



II. *Solution by* Professor SYLVESTER.

Take Op, Oq, Or, the third parts of OP, OQ, OR respectively, and coinciding with them in direction; complete the parallelograms Oi, Oj, ij; then G, the fourth point of the last parallelogram, is the point required, and the system of jointed rods required is given in the figure.

For obviously the projection of OG on any line is equal to the sum of the projections of Op, Oq, Or thereon. Hence G is the centroid of the points P, Q, R, and therefore of the area of the triangle PQR.

This solution shows how to sum mechanically the motions of any number of given points in a linkage; in a word, gives the rule for kinematical addition. Of course the construction can be extended to any number of points so as to give their centroid.

5312. (By the EDITOR.)—If a, b, c be the sides of a triangle; R, r, r_1 r_2, r_3 the radii of the circumscribed, inscribed, and escribed circles; and ρ the radius of the circle inscribed in the orthocentric triangle; prove that

$$4(R^2+2Rr_1)-(a^2+b^2+c^2-bc+ca+ab) = 2(r_1^2-R\rho),$$

with two analogous formulæ in r_2 and r_3.

Solution by Dr. BOOTH, F.R.S.

In BOOTH'S *New Geometrical Methods*, Vol. II., p. 305, eq. (d), it is shown that if Θ be the orthocentre of the triangle, and Ω the centre of one of the escribed circles, $\overline{\Theta\Omega}^2$ is equal to the expression given for reduction;

thus $$2\Theta\Omega = 8R^2+16Rr_1+4bc-2(bc+ac+ab)-2(a^2+b^2+c^2);$$

and in sec. [200], p. 303, it is shown that

$$4R\rho = a^2+b^2+c^2-8R^2;$$

hence, adding together these expressions, we have

$$2\overline{\Theta\Omega}^2 + 4R\rho = 16Rr_1 + 4bc - 4s^2.$$

Substituting for r_1 its value $\frac{sr}{s-a}$, and for $16Rsr$ its value $4abc$,

we have $$2\overline{\Theta\Omega}^2 + 4R\rho = \frac{16Rsr + 4bcs - 4s^3 - 4abc + 4s^2a}{s-a}$$

$$= \frac{4s(s-b)(s-c)}{s-a} = \frac{4s^2(s-b)(s-c)}{s(s-a)} = 4s^2\tan^2\tfrac{1}{2}A = 4r_1^2.$$

Hence $\overline{\Theta\Omega}^2 = 2(r_1^2 - R\rho)$, and like expressions may be found for the centres of the two other escribed circles. [For three other proofs of this property see pp. 85—87 of Vol. XXVII. of *Reprint*. The above question is connected with the *second* of these solutions.]

5319. (By C. Taylor, M.A.)—In the right circular cone, if C, C′ be the centres of a pair of focal spheres, show that the sphere on CC′ contains the auxiliary circle of the section.

Solution by R. F. Davis, B.A.

Let O be the vertex of the cone, and AA′ the major axis of the section. Then C, C′ may be regarded as the centres of the inscribed and escribed circles of the triangle OAA′. Hence E, the middle point of CC′, lies on the circle circumscribing the triangle OAA′; and EC = EA, &c.

[In Dr. Booth's *New Geometrical Methods* (Vol. ii., p. 391, Fig. 72), it is shown that the sphere whose diameter is the line that joins the centres of the *focal spheres*—which Dr. Booth calls the *central sphere*—passes through the ends of the major axis of the conic section; hence the plane of the conic cuts the *central sphere* in the auxiliary circle.]

5226. (By S. Tebay, B.A.)—A person has a rectangular plate of metal, length a and breadth b inches, valued at m shillings a square inch. Equal squares are cut from the corners, and the ends and sides turned up to form a vessel, which is filled with a material at n shillings a cubic inch. If the vessel be formed at random, find the average value of the box and contents; if with the greatest skill, find the depth of the vessel.

Solution by A. MARTIN; *the* PROPOSER; *and others.*

Let x be a side of the square cut from each corner; then $4mx^2$ is the value cut from the plate, $mab-4mx^2$ the value of the box, $nx(a-2x)(b-2x)$ the value of its contents, and $mab+nabx-2\{2m+n(a+b)\}x^2+4nx^3$ $(=V$, say) the value of box and contents. Hence the average value is

$$\frac{\int_0^{\frac{1}{2}b} V\,dx}{\int_0^{\frac{1}{2}b} dx} = \frac{2}{b}\int_0^{\frac{1}{2}b} V\,dx = \tfrac{1}{2}mb(3a-b)+\tfrac{1}{24}nb^2(2a-b).$$

When formed with the greatest skill, the value of the box and its contents will be a maximum; and then $\frac{dV}{dx}=0$ gives

$$x=\tfrac{1}{6}\left[\frac{2m}{n}+(a+b)-\left\{\frac{4m^2}{n^2}+\frac{4m}{n}(a+b)+(a^2-ab+b^2)\right\}^{\frac{1}{2}}\right].$$

The *negative* sign of the radical makes $\frac{d^2V}{dx^2}$ negative, and gives the maximum value required.

4830. (By T. COTTERILL, M.A.)—Let a function of any letters be called alternating which changes its sign only, when any two of the letters are transposed, and be denoted by the letters being placed within parentheses. If the function is of such a nature that the equation

$$(pxy)(pzt)+(pyz)(pxt)+(pzx)(pyt)=0$$

holds for any values of letters, then the expression

$$\{(acd)(aef)(bde)(bcf)-(bcd)(bef)(ade)(acf)\}$$

is alternating, and including itself can be put under fifteen different forms.

Solution by J. HAMMOND, B.A.

Putting, for p, a and b successively; and for x, y, z, t putting c, d, e, f respectively, the equation becomes

$$(acd)(aef)+(ade)(acf)+(aec)(adf)=0,$$

and $$(bcd)(bef)+(bde)(bcf)+(bec)(bdf)=0 \text{ successively.}$$

Multiplying the first by $(bde)(bcf)$ and the second by $(ade)(acf)$ and subtracting, we get

$$\{(acd)(aef)(bde)(bcf)-(bcd)(bef)(ade)(acf)\}$$
$$+\{(aec)(adf)(bde)(bef)-(bec)(bdf)(ade)(acf)\}=0;$$

or, denoting the first of these expressions by $F(a, b, c, d, e, f)$,

$$F(a, b, c, d, e, f) = -\{(ace)(adf)(bed)(bcf) - (bcd)(bef)(aed)(acf)\}$$
$$= -F(a, b, c, e, d, f).$$

Whence it is evident that the effect of transposing any two of the letters c, d, e, f changes the sign of the expression.

Again $F(a, b, c, d, e, f)$ may be written in the form

$$\{(cad)(cbf)(edb)(eaf) - (ead)(ebf)(cdb)(caf)\},$$

and it is seen that a change between any of the letters a, b, d, f changes the sign of the expression.

Thus the expression $F(a, b, c, d, e, f)$ is alternating, and may be put under as many different forms as there are different combinations of 4 letters, that is, 15 forms.

5148. (By the Rev. J. Blissard, B.A.)—Prove that

$$1^6 - 2^6 + 3^6 - \ldots \pm x^6 = \tfrac{1}{2}(-1)^{x+1}(x^6 + 3x^5 - 5x^3 + 3x) \ldots\ldots\ldots (1);$$

$$(1.2.3)^2 - (2.3.4)^2 + \ldots \pm \{x(x+1)(x+2)\}^2 =$$
$$\tfrac{1}{2}(-1)^{x+1}(x^6 + 9x^5 + 28x^4 + 33x^3 + 7x^2 - 6x) \ldots\ldots (2).$$

Solution by J. Hammond, L. W. Jones, *and others.*

Since $$F(E)\, a^x \phi(x) = a^x F(aE)\, \phi(x),$$

$$\Sigma(-)^{x+1}\phi(x) = (-)^{x+1}\frac{1}{-E-1}\phi(x) = (-)^x \frac{1}{e^D+1}\phi(x)$$

$$= (-)^x \left\{\frac{1}{2} - \frac{(2^2-1)B_1}{\lfloor 2}D + \frac{(2^4-1)B_2}{\lfloor 4}D^3 - \frac{(2^6-1)B_3}{\lfloor 6}D^5 + \&c.\right\}\phi(x)$$

$$= (-)^x \left\{\tfrac{1}{2} - \tfrac{1}{4}D + \frac{1}{2\lfloor 4}D^3 - \frac{3}{2\lfloor 6}D^5 + \&c.\right\}\phi(x).$$

Now $$\phi(1) - \phi(2) + \phi(3) - \ldots + (-)^{x+1}\phi(x)$$
$$= (-)^{x+1}\phi(x) + \Sigma(-)^{x+1}\phi(x) + \text{const.}$$
$$= \tfrac{1}{2}(-)^{x+1}\left\{\phi(x) + \tfrac{1}{2}\phi'(x) - \frac{1}{\lfloor 4}\phi'''(x) + \frac{3}{\lfloor 6}\phi^{v}(x) - \&c.\right\} + \text{const.}$$

1. Put $\phi(x) = x^6$, then we have

$$\text{Series} = \tfrac{1}{2}(-)^{x+1}(x^6 + 3x^5 - 5x^3 + 3x),$$

the constant being found from the condition that the series vanishes when $x = 0$.

2. Put $\varphi(x) = \{x(x+1)(x+2)\}^2 = x^6+6x^5+13x^4+12x^3+4x^2$;

then, determining the constant as before, we have

$$\begin{aligned}\text{Series} &= \tfrac{1}{2}(-)^{x+1}\{x^6+6x^5+13x^4+12x^3+\ 4x^2\\ &\qquad +3x^5+15x^4+26x^3+18x^2+\ 4x\\ &\qquad\qquad -\ 5x^3-15x^2-13x-3\\ &\qquad\qquad\qquad +\ 3x+3\}\\ &= \tfrac{1}{2}(-)^{x+1}\{x^6+9x^5+28x^4+33x^3+7x^2-6x\}.\end{aligned}$$

5294. (By J. L. McKenzie, B.A.)—Through a fixed point A on a circle three fixed lines are drawn, meeting the circle in $B_1B_2B_3$. A line is drawn through B_1, parallel to a fixed direction, and meeting the circle in C_1; and through C_1 a line S_1 is drawn parallel to B_2B_3. Prove that the three lines S obtained by permuting $B_1B_2B_3$ intersect in a point on the circle.

Solution by R. F. Davis, B.A.

Let S_1 meet the circle again in P_1, and the six points be ranged on the circumference in the order $AB_3B_2B_1C_1P_1$. Then, denoting by (ab_1) the arc AB_1 measured in the above standard direction of rotation, since A is a fixed point and B_1C_1 a fixed direction, $(ab_1) = (c_1a) + \text{constant}$. Also, since C_1P_1 and B_2B_3 are parallel, $(p_1a) + (ab_3) = (b_2c_1)$. Hence we have

$(p_1a) + (ab_1) + (ab_2) + (ab_3) = (ab_2) + (b_2c_1) + (c_1a) + \text{const.} = \text{circumf.} + \text{const.}$

Thus the value of (p_1a) is symmetrical with respect to $b_1b_2b_3$, and hence &c.

5076. (By the Rev. A. F. Torry, M.A.)—A chord PQ cuts off a constant area from a given oval curve; show that the radius of curvature of its envelop will be $\frac{1}{4}PQ(\cot\theta + \cot\phi)$, where θ and ϕ are the angles at which PQ cuts the given curve.

Solution by E. B. Elliott, M.A.

PQ is bisected at the point R, where it meets its envelope, for $\frac{1}{2}(RP^2 \sim Ra^2)\,d\psi$, the increment of area due to varying its position through an angle $d\psi$, vanishes. Let now R′ be the middle point of the consecutive position P′R′A′, then RR′ is the elemental arc of the envelope.

Now $RP' = RP\{1 + d\psi \cot\theta\}$, and $RA' = RA\{1 - d\psi \cot\phi\}$;

therefore $RR' = \frac{1}{2}(RP' - RQ') = \frac{1}{4} PQ\,(\cot\theta + \cot\phi)\,d\psi$,

since $RP = RQ = \frac{1}{2} PQ$;

therefore $$\rho = \frac{RR'}{d\psi} = \frac{PQ}{4}(\cot\theta + \cot\phi).$$

5203. (By R. Tucker, M.A.)—Three particles are projected in the same vertical plane with the same velocity, but at different elevations: prove that the triangle formed by joining the foci is double that formed by joining the vertices of the parabolic paths.

Solution by R. E. Riley, B.A.; S. Forde, M.A.; *and others.*

Let v = velocity of projection, α_1, α_2, α_3 the angles, and take for axes of x and y the horizontal and vertical through the point of projection. Then the coordinates of the focus and vertex of the path (α_1) are respectively

$$x = \frac{v^2 \sin 2\alpha_1}{2g},\quad y = \frac{v^2}{2g}(2\sin^2\alpha_1 - 1);\quad x' = \frac{v^2 \sin 2\alpha_1}{2g},\quad y' = \frac{v^2}{2g}(\sin^2\alpha_1).$$

Hence the area of the vertex-triangle varies as

$$(\sin^2\alpha_1 + \sin^2\alpha_2)(\sin 2\alpha_2 - \sin 2\alpha_1) + \text{two similar terms};$$

and the area of the focus-triangle varies as

$$2(\sin^2\alpha_1 + \sin^2\alpha_2 - 1)(\sin 2\alpha_2 - \sin 2\alpha_1) + \text{two similar terms},$$

$$= \text{twice the former area}$$

$$- 2(\sin 2\alpha_2 - \sin 2\alpha_1 + \sin 2\alpha_3 - \sin 2\alpha_2 + \sin 2\alpha_1 - \sin 2\alpha_3),$$

$$= \text{twice the former, or vertex-area.}$$

A New Method of Solving Equations of the Form of $x^2 - Ay^2 = 1$.

By Dr. Hart.

Let $A = r^2 \pm m$; then $x^2 - (r^2 \pm m)y^2 = 1$; hence we have

$$x^2 - r^2y^2 = 1 \pm my^2,\quad \text{or}\quad (x + ry)(x - ry) = (1 \pm my^2) \times 1.$$

Put $x + ry = 1 \pm my^2$, and $x - ry = 1$; then we have

$$x = \frac{2 \pm my^2}{2},\ \text{and}\ ry = \frac{\pm my^2}{2};\ \text{therefore}\ y = \frac{2r}{\pm m},\ \text{whence}\ x = \frac{2r^2 \pm m}{\pm m}.$$

In these general values of x and y, r may be any number.

When r is a prime number, then $m = \pm 1, \pm 2, \pm r$, and $\pm 2r$, thus giving 8 different values of A, and as many of x, but only 4 different values of y.

When r is the product of two prime numbers (a, b), then $m = \pm 1, \pm 2, \pm a, \pm b, \pm 2a, \pm 2b, \pm ab$, and $\pm 2ab$, thus giving 16 different values of A and x, and 8 of y.

When r is the product of three prime factors a, b, c, it can be shown that there are 32 different values of A, x, and 16 of y, and so on for $4 . 5 . 6 \ldots n$ factors.

When r is the product of factors, some of which are prime and others composite, we shall also have an infinite number of values of A, x, y. It should be observed that, when m is such that $r^2 \pm m = \square$, a solution is impossible. A large portion of the tables constructed by Legendre, Degen, and others could, by the simple formulæ for A, x, y given above, have been much more easily formed than by the common method. But there is a large portion of these tables that cannot be found by the above formulæ, for finding which there is a new method; and this method, together with a method of solving the equation $x^2 - Ay^2 = -1$, I hope to give at a future time.

5308. (By Professor Townsend, F.R.S.)—A slender circular cord, of uniform thickness and arbitrary length, being supposed to intersect at right angles, in a plane perpendicular to its axis, a thin circular tube, of arbitrary radius and infinite length, repelling it according to the law of the inverse fourth power of the distance; show that the cord will, without deformation at any point, be held in free equilibrium by the repulsion of the tube.

I. *Solution by* Colonel A. R. Clarke, C.B., F.R.S.

Let O be the centre of the circular section of the cylinder by the plane of the string; Q any point of this circle; P, P′ a point within and another without the circle in the straight line OPP′. From Q draw QM perpendicular to OP, and let Z be any point in the generating line QZ of the cylinder. The repulsive force exercised by the element dm at Z on P or P′, resolved in the direction PO towards the centre of the cylinder, is

$$\frac{dm \,.\, \mathrm{PM}}{\mathrm{PZ}^5}, \quad \text{or} \quad \frac{dm \,.\, \mathrm{P'M}}{\mathrm{P'Z}^5}.$$

Let $\mathrm{POQ} = \theta$, $\mathrm{OQ} = a$, $\mathrm{OP} = c$; then the repulsion of the entire cylinder at P or P′ in the direction PO is

$$= 4 \int_0^{\pi} \int_0^{\infty} \frac{a(a \cos\theta - c)\, d\theta\, dz}{(z^2 + a^2 + c^2 - 2ac\cos\theta)^{\frac{5}{2}}} = \tfrac{8}{3} a \int_0^{\pi} \frac{(a\cos\theta - c)\, d\theta}{(a^2 + c^2 - 2ac\cos\theta)^2}$$

$$= \tfrac{8}{3} a \frac{d}{dc} \int_0^{\pi} \frac{\frac{1}{2} d\theta}{a^2 + c^2 - 2ac\cos\theta},$$

which, according as c is greater or less than a, is

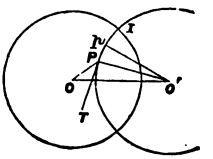
$$= \tfrac{8}{3}a\frac{d}{dc}\left[\frac{1}{c^2-a^2}\tan^{-1}\frac{a+c}{c-a}\tan\tfrac{1}{2}\theta\right]_0^\pi = \tfrac{4}{3}a\pi\frac{d}{dc}\left(\frac{1}{c^2-a^2}\right),$$

or $$= \tfrac{8}{3}a\frac{d}{dc}\left[\frac{1}{a^2-c^2}\tan^{-1}\frac{a+c}{a-c}\tan\tfrac{1}{2}\theta\right]_0^\pi = \tfrac{4}{3}a\pi\frac{d}{dc}\left(\frac{1}{a^2-c^2}\right).$$

Hence the expression for the action of the cylinder towards its centre is

$$\text{at P, } \frac{8\pi}{3}\frac{ac}{(a^2-c^2)^2}; \quad \text{at P', } -\frac{8\pi}{3}\frac{ac}{(c^2-a^2)^2}.$$

Now consider the effect of these forces on the string. Let O' be the centre of the circle formed by the string; P any point in it, *within* the cylinder; $OP=r$, $O'P=k=(c^2-a^2)^{\frac{1}{2}}$, $OO'=c$. At P draw the tangent PT; then



$$\sin OPT = -\cos OPO' = \frac{O'O^2-O'P^2-OP^2}{2PO\,PO'} = \frac{a^2-r^2}{2rk};$$

and if we put for the force at P $\frac{\mu r}{(a^2-r^2)^2}$, the component in the direction O'P normal to the string at P is $\frac{\mu}{2k}\frac{1}{a^2-r^2}$, which is positive. Let p be a point of the string indefinitely near to P, and $PO'p=d\theta$; then, if m be the mass of unit length of the string, that of $Pp=mkd\theta$; and, t denoting the tension of the string, the forces which act on pP in the line O'P are

$$\frac{\mu}{2k}\cdot\frac{mkd\theta}{a^2-r^2} = td\theta; \quad \text{therefore} \quad t=\frac{m}{2}\cdot\frac{1}{a^2-r^2}.$$

It remains only to show that this tension is in equilibrium with the tangential component of the force in PO. This component

$$-\frac{\mu rm}{(a^2-r^2)^2}kd\theta\cos OPT = \frac{\mu mr\,dr}{(a^2-r^2)^2}.$$

This should be equal to dt, and it is so obviously. Hence the portion of the string within the cylinder is in equilibrium, and the same may be proved of the part which is outside the cylinder. For the equilibrium of the string at either of the points, as I, where it intersects normally the cylinder, it is only necessary to observe that, if J, J' be points of the string equidistant from and infinitely near to I, one being within and the other without the cylinder, the tensions at J, J' are equal and the forces equal and opposite, as may very easily be seen.

Otherwise, from the law of force, we may obtain directly the equation of the thread in the form $p=c(a^2-r^2)$, which is that of any circle cutting the cylinder at right angles.

II. *Solution by the* PROPOSER.

The repulsion of the tube at any point, external or internal to its surface, being, as may be easily shewn, the same, for the law of the inverse

fourth power of the distance, as if four-thirds of the mass of its unit of length were distributed, by uniform longitudinal condensation all through, over the annular area of its transverse section by the plane through the point perpendicular to its axis, and repelled, with the same absolute force, according to the law of the inverse cube of the distance instead; and the repulsion of a thin uniform circular ring, for the law of the inverse cube of the distance, at any point external or internal to its mass, varying directly as the linear distance from its centre, and inversely as the fourth power of the tangential distance from its circumference, varies consequently, for every point of a uniform circular cord intersecting its circumference at right angles, directly as the distance from its centre, and inversely as the square of the distance from the polar of its centre with respect to the cord; and, this being the known law for the free equilibrium of such a cord under the action of a central force emanating from any point in its plane, therefore, &c.

5277. (By Professor LLOYD TANNER, M.A.)—A number consists of n digits; its square ends with the same n digits in the same order; show that (1) for any given value of n there are two and only two such numbers in each of the scales of notation whose radices are 6, 10, 12 respectively; and (2) in any other scale (radix not greater than 12) no number (but 1 or 0) exists having this property. [The Proposer remarks that this Question has been suggested by Quest. 5009, *Reprint*, Vol. XXVI., p. 28, and that the two theorems of Quest. 5009 are satisfied by the numbers sought in the present Question. See Solution III., Vol. XXVI., p. 29.]

I. *Solution by* E. B. ELLIOTT, M.A.

If m be a number which when expressed in the scale of r has the property in question, then m^2-m, *i.e.*, $m(m-1)$, is a multiple of r^n, unless it be zero, in which case $m = 0$ or 1.

Now no prime number can divide both m and $m-1$. Thus if r be a prime or a power of a prime, $m(m-1)$ cannot be a multiple of r^n unless either m or $m-1$ is so; that is to say, unless m be a number of at least $n+1$ digits, the last n of which are either all cyphers or $n-1$ cyphers and a unit. Thus, with the exception of 0 and 1, there is no number of n digits whose square ends in the same n digits in either of the scales of 2, 3, 5, 7, 8, 9, 11, 13, 16, 17, 19, 23, 25, 27, 29.

Now suppose the radix r a product of powers of two distinct primes $= p^a q^b$ say. Then $m(m-1)$ can only be made a multiple of r^n, *i.e.*, of $p^{an}q^{bn}$, by taking $m = xp^{an}$, $m-1 = yq^{bn}$ or $m = y'q^{bn}$, $m-1 = x'p^{an}$, where x, y, x', y' are integers so chosen that

$$xp^{an} - yq^{bn} = 1, \quad \text{or} \quad x'p^{an} - y'q^{bn} = -1.$$

Now each of the equations $xp^{an} - yq^{bn} = \pm 1$ can be solved in positive integers by ordinary methods; each being satisfied by one pair of values of x and y less than q^{bn} and p^{an} respectively (being either the numerator and denominator of the continued fraction convergent immediately preceding $\frac{q^{bn}}{p^{an}}$ or the differences between these and q^{bn} and p^{an} respectively),

and neither having more than one pair of such less solutions (for if α, β be the least pair the general form of solutions is $x = \alpha + q^{bn}t$, $y = \beta + p^{an}t$ in each case). Thus one value of m distinct from 0 or 1 is obtained by each of the suppositions $m = xp^{an}$, $m = y'q^{bn}$. In each of the scales then of radius 6, 10, 12, 14, 15, 18, 20, 21, 22, 24, 26, 28, all of which belong to the form p^aq^b, there are two numbers of the kind in question.

The scale of $30 (= 2.3.5)$ is the first instance of another class whose radices have as factors powers of three distinct primes. It is easy to see that in such scales there are six numbers of the kind in question. And generally in any scale whose radii is a product of powers of k distinct primes there are $2^k - 2$.

II. *Solution by the* Proposer.

Let one of the numbers required be $N \equiv p_0 + p_1 r + \ldots + p_{n-1} r^{n-1}$; then, by the question, $N^2 - N = kr^n$, or

$$(p_0 - 1 + p_1 r + \ldots + p_{n-1} r^{n-1})(p_0 + p_1 r + \ldots + p_{n-1} r^{n-1}) = kr^n.$$

Expressing that the digit in the units place is zero, we have

$$p_0^2 - p_0 = m_0 r \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Now this equation is satisfied for all values of r by taking $m_0 = 0$ and $p_0 = 0$ or 1. But in either case it is easily found that $p_1, p_2 \ldots p_{n-1}$ all vanish, so that N becomes simply 0 or 1.

Supposing that n_0 does not vanish, (1) admits of solutions for certain values of r only. Taking the usual scales of notation, viz., r not greater than 12, the following are the only solutions:

$$r = 6,\ p_0 = 3, 4; \quad r = 10,\ p_0 = 5, 6; \quad r = 12,\ p_0 = 4, 9.$$

Having chosen one of these solutions, we get the following equations for p_1, &c., by expressing that the digits in the tens, hundreds, &c. places are all zero,

$$(2p_0 - 1)p_1 + m_0 = m_1 r, \quad (2p_0 - 1)p_2 + p_1^2 + m_1 = m_2 r, \ldots\ldots,$$
$$(2p_0 - 1)p_s + p_{s-1}p_1 + p_{s-2}p_2 + \ldots + p_1 p_{s-1} + m_{s-1} = m_s r.$$

Now each of these equations admits of only one solution; for in each case the multiples of $(2p_0 - 1)$ by 1, 2, ... $r-1$ end in $r-1$ different digits. Thus, knowing m_0, p_1 is determined by the first equation without ambiguity, and m_1 is then known. The second equation gives p_2, m_2 without ambiguity, and so on. There being only two values of p_0 for each of the radices named, there are, therefore, only two values of N when n is specified.

The calculation of the digits presents no difficulty. The values of N (to ten digits each) are

$r = 6$, 3221350213; 3334205344,
$r = 10$, 8212890625; 1787109376,
$r = 12$, 21e61e3854; 9t05t08369 $(t = 9+1,\ e = 9+2)$.

The above results are susceptible of generalization. If q be the number of different prime factors of r, the equation (1) has $q(q-1)$ different solutions (besides 0, 1). Thus, if r be any power of a prime number, *i.e.*, if $q = 1$, there are no numbers of the kind sought. When $q = 2$, there are two values of p_0, as above. When $q = 3$, we have six values of p_0 (*e.g.*, $r = 30$, $p_0 = 6, 10, 15, 16, 21, 25$); and so on. The above number of values of p_0

being even, suggests that these values occur in pairs. This is so, for if $p_0 + p'_0 = r + 1$, and either p_0 or p'_0 be a root of (1), then the other will also satisfy (1).

Trial of some typical examples suggests that $2p_0 - 1$ is always prime to r. If this be so, it would follow as above, that to each value of p_0 corresponds one and only one value of N of any specified number of digits.

On the Arithmetic of Compound Musical Tones.

By Alexander J. Ellis, F.R.S.

1. *Pitch and Simple Tone.*—Let the "pitch" of a "simple" tone mean the number of the pendular vibrations made by the air in one second while it sounds, and let this number, which may be integral, fractional, or, as frequently, incommensurable, be used for the name of the tone. Thus n is a simple tone of the pitch n.

2. *Compound Tone and Partial.* — Let a "compound" or ordinary musical tone or "note" making in one second n compound vibrations, compounded of a theoretically infinite, but practically finite, number of pendular vibrations, each of which answers to a simple tone called for that reason a "partial," be represented by n'. Then, let

$$n' = (n, 2n, 3n, 4n, \ldots),$$

or

$$n' = n(1, 2, 3, 4, \ldots),$$

represent that, when n' is sounded, the effect on the ear is the same as if the several simple tones n, $2n$, $3n$, &c. were sounded simultaneously. In fact, the ear analyses the compound tone n' into such a combination.

3. *Interval.*—Let m' be another compound tone or note $= m\,(1, 2, 3, 4\ldots)$, where m is less than n, then the "interval" m' to n' is considered to be the same as that of m to n, and is generally represented either by $m : n$ or $\frac{n}{m}$ (which are not algebraically equivalent forms, of course). The intervals between the partials of m' and those of n' are not generally considered by the ear, but become of importance in particular cases, and are expressed in the same way.

4. *Beats and Differentials.*—In the lower part of the scale (pitch small) two simple tones m, n, when sounded together, will "beat" when $\frac{n}{m} < \frac{5}{4}$; in the middle parts (pitch moderate) when $\frac{n}{m} < \frac{8}{5}$; and in the upper parts (pitch large) when $\frac{n}{m} < \frac{10}{9}$ or $< \frac{16}{15}$; but beats near the limits either of these fractions or of unity (unison) are not much felt. Theoretically, all the partials must be considered to beat with each other, and the distinctness of the beat to depend upon the pitch of the lower tone m and the magnitude of the fraction $n \div m$. The number of beats in a second is $n - m$. Also, $n - m$ is the pitch of a generated simple tone, here called a "differential." Arithmetically, we must consider that all the partials of a single compound tone m' beat together, and form differentials, and also that all the partials of two compound tones m', n', when sounded toge-

ther, produce beats and differentials, except they be identical in pitch. Practically, only the lower and stronger of the partials within certain distances produce beats, and such differentials as the partials may produce are drowned in the coincident and stronger lower partials.

5. *Harmonics of a Fundamental.*—If we form the compound tones or notes m', $2m'$, $3m'$, &c. (by tuning strings, pipes, or reeds), their lowest partials m, $2m$, $3m$, &c. will be the partials of $m' = m$ (1, 2, 3...). Such "compound tones" are called the "harmonics" of the "fundamental" m', and must be carefully distinguished from its "partials," which are all "simple tones." Practically, the fundamental itself may not exist, because its lower partials are too slow to be heard, as a simple tone of less than 16 is scarcely audible, and even 16 (simple) is heard with difficulty. When 16′ (compound) is supposed to be sounded, we seldom hear more than 16 (2, 3, 4 ...). Observe that the interval between two harmonics of the same fundamental is necessarily expressible as a finite fraction; but that the interval between two harmonics of different fundamentals may not be expressible except by an incommensurable number, as, for example, when they belong to the equally-tempered scale.

6. *Concords or Combinations of Harmonics producing Consonances, Just Intonation.*—If the lower partials of a fundamental m' be audible, and we sound with it any of its harmonics as m' (p, q, r, &c.), we merely "reinforce" the pm, $2pm$, $3pm$, &c., qm, $2qm$, $3qm$, &c., rm, $2rm$, $3rm$, &c., partials of m', and the result is only "a new note," m'' having a different distribution of loudness of partials, producing a different "quality of tone," because all the partials of all the harmonics are also partials of the fundamental.

But when the fundamental is absent, or its lower partials are inaudible, we have for the effect m' (p, q, r) a series of partials in which some of them will reinforce each other; thus pqm and $h \,.\, pqm$ occur on both pm' and qm'; prm and $h \,.\, prm$ on both pm' and rm', and $pqrm$ and $l \,.\, pqrm$ on all three pm', qm', rm', whatever integers be substituted for h, k, l; and other simple tones are supplied by the differentials. The result is usually called a "concord" in both cases.

For example,

Partials	1,	2,	3,	4,	5,	6,	7,	8, &c.
$m'(4, 5, 6) = m$ (	4,	8,	12.,	*16,	20..,	†24...,	‡28,	§32, &c.
	5,	10,	*15,	20..,	†25,	§‡30....,	‖35,	¶40, &c.
	6,	12.,	18,	†24...,	§30....,	‖36,	¶42,	48, &c.)

Here the top line shows the number of the partials in each compound note, whence their relative loudness may be inferred, (.) (..) (...) (....) the reinforcing partials, and * † ‡ § ‖ ¶ the partials which beat sensibly, but as fast as m, $2m$, ... times in a second producing the "rattle" of the partials or "roughness" which characterises combinations. To this we must add at least the differentials formed from the lowest partials m ($5-4=1$, $6-5=1$), which also reinforce each other, and m ($6-4=2$). The other differentials may be neglected from their faintness, but they produce roughness. Thus ($15-4=11$) m beats with both $10m$ and $12m$. It is convenient to arrange the above expression as if it were a single compound tone. To show whence the partials arise, and indicate their relative strength, prefix the number of the partial as an index, and, if they come from a differential, prefix a zero as index, to show that they do not result from partials. The expression then becomes

$$m'(4, 5, 6) = m\,(^{0,0}1,\ ^{0}2,\ ^{1}4,\ ^{1}5,\ ^{1}6,\ ^{2}8,\ ^{2}10,\ ^{2,3}12,\ ^{3}15,\ ^{4}16,\ ^{3}18,\ ^{4,5}20,\ ^{4,6}24,\ ^{5}25,\ ^{7}28,\ ^{5,6}30,\ ^{8}32,\ ^{7}35,\ ^{6}36,\ ^{8}40,\ ^{8}48...).$$

Now this differs materially in effect from an ordinary compound tone m', because the three lowest partials m (1, 2, 3), generally predominant, are either very faint or absent, many very characteristic high partials m (7, 9, 11, 13, 14 ...) are entirely absent, the partials m (4, 5, 6, 8, 12, 15, 18), ordinarily by no means prominent, are as marked as the usual (1, 2, 3) m, or even more so, and such very high partials as m (16, 20, 24, 25 ... 48), ordinarily quite imperceptible, become as prominent as the partials m (4, 5, 6, 7, 8). One consequence is the very distinct "roughness" or "rattle" which would be entirely inaudible in single compound notes $4m'$, $5m'$, $6m'$. The characteristic peculiarity, however, of such combinations of compound tones is that several partials of the individual notes range themselves on "the same seat" as partials of the fundamental. Hence I term them "consident" partials, and such combinations as m ($^{2,3}12$, $^{4,5}20$, $^{5,6}30$) I call "considences," a term which I extend to the whole combination of harmonic compound tones. When notes are tuned in such a way as to produce consident partials, they are said to be in "just," that is "exact," intonation.

7. *Dissidences and Discords.*—Now, instead of combining two harmonics as pm', qm', where p and q are integers, combine pm' and $q(1+t)m'$, where t is small, so that the interval of the altered tones is $q(1+t)m \div qm = 1+t$. Then, whereas we had the considence $^{p,q}pqm$, we shall have two partials ^{p}pqm, $^{q}pq(1+t)m$, which also form the interval $1+t$, and beat $pqtm$ times in a second. Hence, whereas $m'(p, q)$ produced two partials of the separate tones, which were both consident with a partial of the fundamental, the altered form $m'[p, q(1+t)]$ produces, among other, two partials which are nearly but not quite consident with a partial of the fundamental, and which beat together. This "tearing away of the consident partials from their seat" I call "dissidence," the beats thence arising I call "dissident beats," and I mark the occurrence of dissidence by using square brackets in the formula. Since t is always very small in all cases of practice, the "dissident beats" $pqtm$ are much slower than the "rattle," and are readily distinguished by the ear. For the same values of p, q, m, these beats vary as t. Hence, in "tempering," which consists in substituting a dissidence for a considence, for facilitating execution on keyed instruments and modulation generally, the aim is to make t very small. The following table shows the value of t for the principal intervals in the principal temperaments. The larger number in the just ratio must be multiplied by $1+t$ to produce its tempered representative, the lower number remaining unaltered.

NAME OF INTERVAL.	Just Ratios.	VALUES OF t IN THE TEMPERAMENTS OF					
		Helmholtz	Bosanquet	Appunn	Meantone	Equal	Pythagoras
Diatonic Semitone	15 : 16	+·000 141	+·000 853	+·001 130	+·003 110	−·006 799	−·0125
Minor Tone	9 : 10	+·000 282	−·000 833	−·001 130	+·006 230	+·010 216	+·0125
Major Tone	8 : 9	−·000 282	−·000 282	0	−·006 230	+·002 491	0
Minor Third	5 : 6	−·000 141	+·000 776	+·001 130	−·003 110	−·009 076	−·0125
Major Third	4 : 5	0	−·000 813	−·001 130	0	+·007 937	+·0125
Fourth	3 : 4	+·000 141	+·000 039	0	+·003 110	+·001 130	0
Fifth	2 : 3	−·000 141	−·000 039	0	−·003 110	−·001 130	0
Minor Sixth	5 : 8	0	+·000 813	+·001 130	0	−·007 937	−·0125
Major Sixth	3 : 5	+·000 141	−·000 776	−·001 130	+·003 110	+·009 076	+·0125
Minor Seventh	9 : 16	+·000 282	+·000 770	0	+·006 230	+·002 261	0
Acute Mi. Seventh	5 : 9	−·000 282	+·000 730	+·001 130	−·006 230	+·010 216	−·0125
Major Seventh	8 : 15	−·000 141	−·000 853	−·001 130	−·003 100	+·006 799	+·1250

The three first temperaments were proposed by Prof. Helmholtz, Mr.

Bosanquet, and Herr Appunn, the two last of whom have constructed and exhibited harmoniums to shew the musical value of their arrangements as close approximations to just intonation. The first makes every Fifth too flat by $\frac{1}{8}$ skhisma, or m' [2, 3 (1 − ·000141)], and every major Third perfect; the third makes every major Third too flat by a skhisma, or m' [4, 5 (1 − ·001130)], and every Fifth perfect; the second divides the Octave into 53 equal parts, and hence alters every interval, but so slightly that the result is intermediate to the other two. These three temperaments distinguish the major and minor Tone. The last three destroy this distinction. The fourth produces an exact mean between the major and minor Tone, and was used almost exclusively till forty or fifty years ago; it makes every Fifth too flat by a $\frac{1}{4}$ comma, or m' [2, 3 (1 − ·003110)], and every major Third perfect. The fifth divides every Octave into 12 equal parts, and uses them as it best can, altering some intervals almost unbearably, and more closely approaching the sixth, (which was used in Greece for melody only,) than just intonation, which can alone produce considence.

The progressive increase in the absolute values of t shews the progressive deterioration of the temperament. Helmholtz's is the best on paper, but difficult, perhaps impossible, to realise. The easiest to realise are Appunn's and Pythagoras's, because they consist of a series of perfect Fifths; and the next is (at least an approximation to) the mean tone, for which our musical notation is adapted. Mr. Bosanquet's and Equal Temperament are both very difficult to tune accurately.

The effect of the dissident beats must be judged of −(1) by the absolute pitch of the lower beating partial pqm, being worst in low pitches; (2) by the number p, of the beating partials of $q\ (1+t)\ m'$ and pqm' respectively, which determines their loudness, so that for m' [2, 3 $(1+t)$] the effect would be much stronger than for m' [3, 5 $(1+t)$], t remaining constant, and consequently a greater value of t can be endured in the latter case than in the former; (3) by the interval $1+t$ between the beating partials, which however is always sufficiently small to be very disagreeable; (4) by the $pqtm$ numbers of beats produced. Observe, however, that in no dissidence is it possible to produce the effect of a fundamental with a new quality of tone. In all cases we get beats, that is noise, in place of musical tone.

All dissidences ought properly to be considered as "discords," but this term is not applied to those which arise merely from temperament, because such dissidences are the best representations of considences which a tempered scale could furnish. The term "discord" is therefore only applied to those cases in which dissidences would arise even in just intonation by the introduction of notes which are incompatible with consideence. Possibly the acceptance of temperament in lieu of just intonation arises from the inability of the ear to appreciate the intervals between the partials which do not beat, and thus to distinguish between a series of simple tones which would form a compound tone and another which will not. The dissident beats are overlooked from being masked either by evanescent or soft qualities of tone, or by rapidity of execution.

8. *Beats from Differential Tones.*—In dissidences of three notes, we necessarily obtain differential tones even from the lowest partials, which not only will not fit into places as partials of any fundamental, but will sometimes be so close to one another or to some other partial as to produce beats. Thus, if m' (p, q, r) be altered to m' $[p, q\ (1+t), r\ (1+s)]$, the differentials of the former, m $(q-p,\ r-p,\ r-q)$, which fit into the fundamental, become m $[q-p+qt,\ r-p+sr,\ r-q+sr-tq]$, which

cannot possibly fit in and may often beat. Thus, if m' (4, 5, 6) become m' [4, $5+5t$, $6+6s$], the differentials are m [$1+5t$, $2+6s$, $1+6s-5t$], and the first and last of these beat m ($6s-10t$) times in a second. This is quite distinct in both the equal and mean tone temperament, although the beats are between faint notes. But if m' (3, 5, 8) is altered to m' [3 ($1+s$), 5 ($1+t$), 8], the differentials are m [$2+5t-3s$, $5-3s$, $3-5t$], and the two latter beat with the lowest partials m ($5+5t$) and m ($3+3s$) of the principal notes themselves, the beats being $5t+3s$ in each case, so that they reinforce each other and become very conspicuous, making the tempered combination very bad, while the just combination is very good. Similarly for the alteration of m' (2, 3, 5) into m' [2, 3 ($1+s$), 5 ($1+t$)].

9. *Calculations for the Estimation of Considence and Dissidence.*—It may so happen that we do not know of what simple interval any given interval is an alteration. We may then proceed thus, taking as examples dissidences of the form m' [p, q] where p and q are large numbers.

Ex. 1. 1′ [264, 394·772].			Ex. 2. 5·2325′ [50, 63].			Ex. 3. 4′ [64, 81].		
1	264	394·772	1	50	63	1	64	81
2	528	†789·544	2	100	126	2	128	162
3	†792	1184·316	3	150	189	3	192	243
4	1056	‡1579·088	4	200	†252	4	256	†324
5	1320		5	†250	315	5	†320	405
6	‡1584		6	300		6	384	
Dissident partials † (2 3) and ‡ (4, 6). Their interval $1+t = 789·544 \div 792 = ·99689 = 1-·00311$, shewing the dissidence to be a meantone Fifth. Lowest note 264′. Beats $1 \times 2·456$ flat. Differential 130·772.			Dissident partials † (4, 5). Their interval $1+t = 252 \div 250 = 1·008$, shewing the dissidence to be nearly an equal major Third. Lowest note $50 \times 5·2325' = 261·625'$. Beats $2 \times 5·2325 = 10·465$ sharp. Differential $13 \times 5·2325 = 68·0225$.			Dissident partials † (4, 5). Their interval $1+t = 324 \div 320 = 1·0125$, a comma, shewing the dissidence to be a Greek major Third. Lowest note $4 \times 64' = 256'$. Beats $4 \times 4 = 16$ sharp. Differential $4 \times 17 = 68$.		

The following is an explanation of the calculation and results.

Ex. 1. 1′ [264, 394·772], the 1′ shows that the absolute pitches of the dissident notes are given within the brackets. The dissident beats arise from the partials which are nearly alike, here from the 2nd and 3rd partials, which if identical would have determined a Fifth, shewing that we have a disturbed Fifth. As the 2nd partial of the higher note is flatter than the 3rd partial of the lower note, the Fifth is flat. Dividing the pitches of these partials, we necessarily get for $1+t$ a number less than 1, so that t is negative, and $= -·00311$, which is the characteristic of the mean tone Fifth, as stated in Art. 7. The differential is the difference of the first partials of the two notes.

Ex. 2. 5·2325′ [50, 63] gives only a relative pitch, and the multiplier. The dissidence depending on the 4th and 5th partials shews a disturbed major Third. Their interval being 1·008, t is nearly $= ·007937$, its value in the equal major Third, see Art. 7, for which it was clearly intended. The

absolute pitches are 50 and 63 times the constant multiplier 5·2325. The number of dissident beats is 2 times the difference of the dissident partials multipled by the constant number 5·2325. The differential is the difference of the relative pitch of the first partials or 63−50, multiplied by the constant multiplier.

Ex. 3. This is similar to Ex. 2, but here $t = +\cdot0125$, which is a whole comma, and this determines the dissidence to be a Greek major Third, by Art. 7.

10. *Physical Limitation of Considence.*—Recollecting that, in all usable parts of the scale, partials making an interval $< \frac{1}{9}^{\circ}$ will beat very distinctly, it is not necessary to carry out the process beyond 10 partials of the two tones, if we only wish to discover the law of dissidence, and the ear naturally selects as the altered considence one with which it is familiar, as $\frac{2}{1}$, $\frac{3}{2}$, $\frac{4}{3}$, $\frac{5}{4}$, $\frac{6}{5}$, $\frac{8}{5}$, $\frac{9}{5}$. If we proceed beyond these numbers, we necessarily obtain dissidences. It is a good exercise to select any two relatively prime numbers of which the greater is not so much as twice the smaller, and determine of what interval they must be considered as alterations, with the value of t, and, assuming some multiplier which will bring the result within usable pitches, determine the number of beats. Thus ·088′ [2973, 3746] beats with partials 4 and 5, and is hence an altered major Third, and as $t = +\cdot008$, it is meant for an equal major Third, and beats $119 \times \cdot088 = 10\cdot47$ sharps in a second. By this means we learn to feel the physical limitation of considence. There is at first a little difficulty in seeing whether two partials lie within beating distance. The simplest way is to add to the smaller one-tenth of itself, and if it then becomes greater than the larger, the interval will be less than $\frac{1}{10}$, and the simple tones are sure to beat. If it remains less, add another tenth of the original number; and if the result is still less than the larger, the interval is greater than $\frac{2}{3}$, and they will not beat sensibly; if the result is greater than the larger number, but nearly equal to it, the beats may be disregarded. If on adding one-twentieth of the smaller number to itself it becomes greater than the larger, the interval is $< \frac{21}{20}$, and the beats will be very offensive.

Thus, in 28′ [9, 16] we obtain 28′ [^{19}, ^{1}16, ^{2}18, ^{3}27, ^{2}32, ^{4}36, ^{5}45, ^{3}48…]. Now 28 [16, 18] = 56 [8, 9] clearly beat, as we may find by $16 + 1\cdot6 = 17\cdot6$, which is nearly 18, but smaller. But do 27, 32 beat? Form $27 + 2\cdot7 = 29\cdot7$, this is a good deal less than 32, and hence any beat must be faint, proceeding to add another tenth, $29\cdot7 + 2\cdot7 = 32\cdot4$, which is so nearly 32 that any beats must be disregarded. For 45, 48 we have $45 + 4\cdot5 = 49\cdot5$, which is very nearly 48, but larger. The beats will therefore be quite distinct beats. Hence the interval [9, 16] is either a disturbed Octave or disturbed major Sixth, of which the former is more easy to recognize. Every chord and combination in just and tempered intonation has to be thus investigated, and its value determined. It must be recollected, however, that the result gives the isolated, and not the combined value, and that dissidences, independently of temperament, can often be used with excellent effect.

5260. (By W. H. H. Hudson, M.A.)—Prove that the radius of curvature of $au = \phi(\theta)$, at the point $\theta = \alpha$, subtends at the pole the angle

$$\tan^{-1}\left\{\frac{\phi'(\alpha)}{\phi(\alpha)} \cdot \frac{\phi(\alpha)^2 + \phi'(\alpha)^2}{\phi'(\alpha)^2 - \phi(\alpha)\,\phi''(\alpha)}\right\}.$$

Solution by S. ROBERTS, M.A.; L. W. JONES, B.A.; *and others.*

The radii of curvature at corresponding points of a curve and its inverse to any modulus subtend the same angle at the pole. It is immaterial therefore whether we consider the curve $au = \phi(\theta)$ or $ar = \phi(\theta)$ where $u = r^{-1}$. In either case, making use of the formula for the centre of curvature, viz., $X = x - \dfrac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}\dfrac{dy}{dx}$, $\quad Y = y + \dfrac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}$,

and turning the initial vector through the angle α, so that $\theta = \theta' + \alpha$, we have, for $\theta' = 0$, $\dfrac{Y}{X} = \tan$ (angle subtended)

$$= \frac{r^2 + \left(\frac{dr}{d\theta'}\right)^2}{\left(\frac{dr}{d\theta'}\right)^2 - r\frac{d^2r}{d\theta'^2}}\frac{\frac{dr}{d\theta'}}{r} = \frac{u^2 + \left(\frac{du}{d\theta'}\right)^2}{\left(\frac{du}{d\theta'}\right)^2 - u\frac{d^2u}{d\theta'^2}}\frac{\frac{du}{d\theta'}}{u}$$

$$= \frac{\phi'(\alpha)}{\phi(\alpha)} \cdot \frac{\phi(\alpha)^2 + (\phi'\alpha)^2}{\phi'(\alpha)^2 - \phi(\alpha)\phi''(\alpha)}.$$

5118. (By R. F. SCOTT, B.A.)—If any binary quantic be transformed by means of a linear substitution, prove that any invariant of the transformed quantic is equal to its value for the original quantic multiplied by a power of the determinant of the substitution equal to the weight of the invariant.

I. *Solution by* J. J. WALKER, M.A.

A proof of the fact stated in this question is contained implicitly in that for the weight of the invariant of a binary quantic. But the statement may be extended to linear transformations of quantics in any number of variables, for which the case of three will serve. Let the coefficient of any term of a ternary quantic be written $a_{\alpha\beta\gamma}$, where α, β, γ are the indices of x, y, z respectively in the term, so that $\alpha + \beta + \gamma = n$. Then, if the order of any invariant be θ, and $a_{\alpha\beta\gamma}\, a_{\alpha'\beta'\gamma'}\, a_{\alpha''\beta''\gamma''} \ldots\ldots$ any term, if the transformation be λx for x only, the determinant will be simply λ, and the multiplier λ^p, where $p = \alpha + \alpha' + \alpha'' \ldots\ldots$. Again, supposing the successive transformations μy for y only, and νz for z only, there will result

also $\qquad p = \beta + \beta' + \beta'' \ldots\ldots, \quad p = \gamma + \gamma' + \gamma'' \ldots\ldots.$

Hence $\qquad 3p = \alpha + \beta + \gamma + \alpha' + \beta' + \gamma' + \alpha'' + \beta'' + \gamma'' \ldots\ldots = n\theta;$

and $p = \frac{1}{3}n\theta =$ weight of invariant as estimated by the sum of any suffixes of the same letter.

II. *Solution by the* PROPOSER.

Let us consider the linear substitution

$$x = \lambda \,.\, X + \,.\,,$$
$$y = \quad . \quad + Y.$$

The determinant of the substitution is λ.

The transformed quantic differs from the original solely in the fact that each coefficient is multiplied by a power of λ equal to the power of n belonging to that coefficient; that is to say, each coefficient is multiplied by a power of λ equal to its weight.

Hence the invariant of the transformed quantic must be equal to the invariant of the original quantic multiplied by a power of λ equal to its weight. This proves the theorem.

5247. (By Professor LLOYD TANNER, M.A.)—If the determinant $\begin{vmatrix} a_1, & a_2, & \dots & a_n \\ b_1, & b_2, & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}$ be represented by $|\ |$; and $\begin{vmatrix} \iota, & \lambda, & \dots \\ j, & l, & \dots \end{vmatrix}$ represent the result of replacing the j^{th}, l^{th}, ... columns of $|\ |$ by $\alpha_\iota, \beta_\iota, \dots, \alpha_\lambda, \beta_\lambda, \dots, \dots$;

prove that $\begin{vmatrix} \begin{vmatrix} \iota \\ j \end{vmatrix}, & \begin{vmatrix} \iota \\ l \end{vmatrix}, & \dots\dots \\ \begin{vmatrix} \lambda \\ j \end{vmatrix}, & \begin{vmatrix} \lambda \\ l \end{vmatrix}, & \dots\dots \\ \dots & \dots & \dots\dots \\ \dots & \dots & \dots\dots \end{vmatrix} = \left\{ \left| \ \right| \right\}^{r-1} \times \begin{vmatrix} \iota, & \lambda, & \dots \\ j, & l, & \dots \end{vmatrix}$,

the determinant on the left being of the r^{th} order.

I. *Solution by the* PROPOSER.

Consider the system of equations

$$\left\| \begin{matrix} \xi_1, \ \xi_2, \ \dots\dots \ \xi_\nu, & x_1, \ x_2, \ \dots\dots \ x_n \\ \alpha_1, \dots \dots\dots \ \alpha_\nu, & \alpha_1, \ \dots \ \dots\dots \ \alpha_n \\ \beta_1, \dots \dots\dots \ \beta_\nu, & b_1, \ \dots \ \dots\dots \ b_n \end{matrix} \right\| = 0 \quad \dots\dots\dots\dots\dots (1).$$

$(n+1$ rows$)$

Taking each of the first ν columns in turn, with the last n columns we form ν equations

$$\left.\begin{matrix} \dots & \dots & \dots & \dots & \dots & \dots \\ \xi_\iota \times \left| \ \right| - \dots - x_j \begin{vmatrix} \iota \\ j \end{vmatrix} - \dots - x_l \begin{vmatrix} \iota \\ l \end{vmatrix} - \dots\dots = 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \xi_\lambda \times \left| \ \right| - \dots - x_j \begin{vmatrix} \lambda \\ j \end{vmatrix} - \dots - x_l \begin{vmatrix} \lambda \\ l \end{vmatrix} - \dots\dots = 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \end{matrix}\right\} \quad \dots\dots\dots\dots\dots (2).$$

Now, assuming that $||$ does not vanish, we have here ν independent equations, for each contains a ξ not in any of the others. Moreover, on the same assumption, when the equations (2) are satisfied, every determinant of the matrix in (1) vanishes, so that (1) does not contain any equations save what can be derived from (2).

Now eliminate $x_l, x_m, \ldots$ from the r equations (2) which contain $\xi_\iota, \xi_\lambda, \xi_\mu, \ldots$. The result is

$$\left|\begin{matrix} \left|\begin{matrix}\lambda\\ l\end{matrix}\right|, & \left|\begin{matrix}\lambda\\ m\end{matrix}\right|, & \ldots \\ \left|\begin{matrix}\mu\\ l\end{matrix}\right|, & \left|\begin{matrix}\mu\\ m\end{matrix}\right|, & \ldots \\ \ldots & \ldots & \ldots \end{matrix}\right| \times \left\{\xi_\iota \left|\begin{matrix} \left|\begin{matrix}\lambda\\ l\end{matrix}\right|, & \left|\begin{matrix}\lambda\\ m\end{matrix}\right|, & \ldots \\ \left|\begin{matrix}\mu\\ l\end{matrix}\right|, & \left|\begin{matrix}\mu\\ m\end{matrix}\right|, & \ldots \\ \ldots & \ldots & \ldots \end{matrix}\right| - \&c.\right\} - \ldots - x_j \left|\begin{matrix} \left|\begin{matrix}\iota\\ j\end{matrix}\right|, & \left|\begin{matrix}\iota\\ l\end{matrix}\right|, & \left|\begin{matrix}\iota\\ m\end{matrix}\right|, & \ldots \\ \left|\begin{matrix}\lambda\\ j\end{matrix}\right|, & \left|\begin{matrix}\lambda\\ l\end{matrix}\right|, & \left|\begin{matrix}\lambda\\ m\end{matrix}\right|, & \ldots \\ \left|\begin{matrix}\mu\\ j\end{matrix}\right|, & \left|\begin{matrix}\mu\\ l\end{matrix}\right|, & \left|\begin{matrix}\mu\\ m\end{matrix}\right|, & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{matrix}\right| - \&c. = 0 \quad \ldots(3).$$

But from (1) we can form an equation containing $\xi_\iota, \xi_\lambda, \xi_\mu, \ldots$, and not containing $x_l, x_m, \ldots$; viz., this is

$$\xi_\iota \left|\begin{matrix}\lambda, & \mu, & \ldots \\ l, & m, & \ldots\end{matrix}\right| - \&c. - x_j \left|\begin{matrix}\iota, & \lambda, & \mu, & \ldots \\ j, & l, & m, & \ldots\end{matrix}\right| - \&c. = 0 \quad \ldots\ldots\ldots(4).$$

And this must be equivalent to (3), since otherwise we should have an equation of (1) not derivable from (2).

Suppose now that the theorem is true for $r-1$; that is, let

$$\left|\begin{matrix} \left|\begin{matrix}\lambda\\ l\end{matrix}\right|, & \left|\begin{matrix}\lambda\\ m\end{matrix}\right|, & \ldots \\ \left|\begin{matrix}\mu\\ l\end{matrix}\right|, & \left|\begin{matrix}\mu\\ m\end{matrix}\right|, & \ldots \\ \ldots & \ldots & \ldots \end{matrix}\right| = \left\{\left|\begin{matrix}\,\\ \,\end{matrix}\right|\right\}^{r-2} \times \left|\begin{matrix}\lambda, & \mu, & \ldots \\ l, & m, & \ldots\end{matrix}\right|.$$

Then, substituting in (3), and comparing the result with (4), we have

$$\left|\begin{matrix} \left|\begin{matrix}\iota\\ j\end{matrix}\right|, & \left|\begin{matrix}\iota\\ l\end{matrix}\right|, & \left|\begin{matrix}\iota\\ m\end{matrix}\right|, & \ldots \\ \left|\begin{matrix}\lambda\\ j\end{matrix}\right|, & \left|\begin{matrix}\lambda\\ l\end{matrix}\right|, & \left|\begin{matrix}\lambda\\ m\end{matrix}\right|, & \ldots \\ \left|\begin{matrix}\mu\\ j\end{matrix}\right|, & \left|\begin{matrix}\mu\\ l\end{matrix}\right|, & \left|\begin{matrix}\mu\\ m\end{matrix}\right|, & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{matrix}\right| = \left\{\left|\begin{matrix}\,\\ \,\end{matrix}\right|\right\}^{r-1} \times \left|\begin{matrix}\iota, & \lambda, & \mu, & \ldots \\ j, & l, & m, & \ldots\end{matrix}\right|.$$

Thus, if the theorem is true for $r-1$, it is true for r. Now the theorem is obviously true when $r=1$, for it becomes

$$\left|\begin{matrix}\iota\\ j\end{matrix}\right| = \left|\begin{matrix}\iota\\ j\end{matrix}\right|;$$

hence it is generally true.

II. *Solution by* G. S. Carr.

Writing the determinants for the case in which $r = 3$ (though the proof will be perfectly general), and denoting as usual by A_j the first

minor of the constituent a_j; the determinant on the left is equivalent to

$$\begin{vmatrix} a_i A_j + \beta_i B_j + \ldots, & a_i A_l + \beta_i B_l + \ldots, & a_i A_m + \beta_i B_m \\ a_\lambda A_j + \beta_\lambda B_j + \ldots, & a_\lambda A_l + \beta_\lambda B_l + \ldots, & a_\lambda A_m + \beta_\lambda B_m \\ a_\mu A_j + \beta_\mu B_j + \ldots, & a_\mu A_l + \beta_\mu B_l + \ldots, & a_\mu A_m + \beta_\mu B_m \end{vmatrix}$$

a determinant of the rth order, each constituent of which is the sum of n terms. Of the elementary determinants of the same order into which this may be resolved, those which have two or more columns identical vanish. The remaining ones constitute, in the first place, $\lfloor\underline{r}$ terms of which the first is

$$A_j B_l C_m \begin{vmatrix} a_i & \beta_i & \gamma_i \\ a_\lambda & \beta_\lambda & \gamma_\lambda \\ a_\mu & \beta_\mu & \gamma_\mu \end{vmatrix},$$

and the sum of these terms forms the product

$$\begin{vmatrix} A_j & A_l & A_m \\ B_j & B_l & B_m \\ C_j & C_l & C_m \end{vmatrix} \begin{vmatrix} a_i & a_\lambda & a_\mu \\ \beta_i & \beta_\lambda & \beta_\mu \\ \gamma_i & \gamma_\lambda & \gamma_\mu \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and the rest form similar products involving the permutations of $\alpha\beta\gamma\delta$, &c. taken r at a time with their corresponding minors.

Now the partial reciprocal of the original determinant of the order r which forms the left factor of (1) is equal to the $r-1^{\text{th}}$ powers of the original determinant multiplied by the complementary of the right factor (Salmon's *Higher Algebra*, p. 17), if the Greek letters be substituted for the English in the stated r columns of the original determinant.

Consequently the sum of all the products like (1) becomes equal to the product of the $r-1^{\text{th}}$ power of the original determinant, and the determinant with the r altered columns, as was to be shewn.

5163. (By A. W. Panton, M.A.)—If the coordinates (x, y, z) of a point on a unicursal cubic are proportional to $a_1\theta^3+b_1\theta^2+c_1\theta+d_1$, $a_2\theta^3+b_2\theta^2+c_2\theta+d_2$, $a_3\theta^3+b_3\theta^2+c_3\theta+d_3$; show that the values of θ at the points of inflexion are the roots of the equation

$$\begin{vmatrix} 1 & -3\theta & 3\theta^2 & -\theta^3 \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix} = 0.$$

[The quadratic that gives the values of θ at the double point is the Hessian of this cubic. See Professor Wolstenholme's solution of Quest. 5080, *Reprint*, Vol. XXVI., pp. 83—86.]

I. *Solution by* J. J. Walker, M.A.

If $\alpha x+\beta y+\gamma z=0$ is any transversal, the three points in which it meets the cubic will be determined by

$$(a_1\alpha+a_2\beta+a_3\gamma)\theta^3 + (b_1\alpha+b_2\beta+b_3\gamma)\theta^2 + (c_1\alpha+c_2\beta+c_3\gamma)\theta + d_1\alpha+d_2\beta+d_3\gamma = 0;$$

and if θ_1 is the value of θ at a point of inflexion, the coefficients of the above equation must be proportional to those of

$$\theta^3 - 3\theta_1\theta^2 + 3\theta_1^2\theta - \theta_1^3 = 0,$$

that is, $\quad a_1\alpha + a_2\beta + a_3\gamma + \lambda = 0, \quad b_1\alpha + b_2\beta + b_3\gamma - 3\theta_1\lambda = 0,$

$$c_1\alpha + c_2\beta + c_3\gamma + 3\theta_1^2\lambda = 0, \quad d_1\alpha + d_2\beta + d_3\gamma - \theta_1^3\lambda = 0.$$

Eliminating α, β, γ, λ, there results the condition that θ_1 must satisfy the equation given in the question.

It is obvious how the equations of the tangents at the points of inflexion, and of the axis of inflexion, may be formed.

II. *Solution by* Lionel H. Rosenthal.

Let $\lambda x + \mu y + \nu z = 0$ be the equation of any line intersecting the curve, then for the parameters of its points of intersection we have the equation

$$\theta^3(a_1\lambda + a_2\mu + a_3\nu) + \theta^2(b_1\lambda + b_2\mu + b_3\nu) + \theta(c_1\lambda + c_2\mu + c_3\nu) + d_1\lambda + d_2\mu - d_3\nu = 0.$$

Hence, if the roots of this equation be θ_1, θ_2, θ_3, we have

$$\begin{aligned}
\Sigma(\theta_1) \,.\, (a_1\lambda + a_2\mu + a_3\nu) + b_1\lambda + b_2\mu + b_3\nu &= 0,\\
\Sigma(\theta_1\theta_2) \,.\, (a_1\lambda + a_2\mu + a_3\nu) - c_1\lambda - c_2\mu - c_3\nu &= 0,\\
\theta_1\theta_2\theta_3 \,.\, (a_1\lambda + a_2\mu + a_3\nu) + d_1\lambda + d_2\mu + d_3\nu &= 0.
\end{aligned}$$

To eliminate λ, μ, ν from these, assume

$$a_1\lambda + a_2\mu + a_3\nu = \Lambda,$$

then we have

$$\begin{aligned}
\Lambda - a_1\lambda - a_2\mu - a_3\nu &= 0,\\
\Lambda \,.\, \Sigma(\theta_1) + b_1\lambda + b_2\mu + b_3\nu &= 0,\\
\Lambda \,.\, \Sigma(\theta_1\theta_2) - c_1\lambda - c_2\mu - c_3\nu &= 0,\\
\Lambda \,.\, \theta_1\theta_2\theta_3 + d_1\lambda + d_2\mu + d_3\nu &= 0;
\end{aligned}$$

hence

$$\begin{vmatrix}
1 & \Sigma(\theta_1) & \Sigma(\theta_1\theta_2) & \theta_1\theta_2\theta_3\\
-a_1 & b_1 & -c_1 & d_1\\
-a_2 & b_2 & -c_2 & d_2\\
-a_3 & b_3 & -c_3 & d_3
\end{vmatrix} = 0.$$

Now, let $\lambda x + \mu y + \nu z = 0$ be an inflexional tangent, and since in that case $\theta_1 = \theta_2 = \theta_3$, we get for the points of inflexion the cubic

$$\begin{vmatrix}
1 & 3\theta & 3\theta^2 & \theta^3\\
-a_1 & b_1 & -c_1 & d_1\\
-a_2 & b_2 & -c_2 & d_2\\
-a_3 & b_3 & -c_3 & d_3
\end{vmatrix} = 0,$$

i.e.,

$$D\theta^3 + 3C\theta^2 + 3B\theta + A = 0.$$

The inflexional tangents are got by substituting the roots of the above equation, in the following, viz.:

$$\{(a_2b_3)\theta^2 + (a_2c_3)\theta + (b_2c_3)\}x + \{(a_3b_1)\theta^2 + (a_3c_1)\theta + (b_3c_1)\}y + \{(a_1b_2)\theta^2 + (a_1c_2)\theta + (b_1c_2)\}z = 0.$$

To find the double point; since each of the tangents at this point meets the curve in two consecutive points on one branch and one point on the other, we have therefore the two relations,

$$\begin{vmatrix} 1 & 2\theta_1+\theta_2 & 2\theta_1\theta_2+\theta_1^2 & \theta_1^2\theta_2 \\ -a_1 & b_1 & -c_1 & d_1 \\ -a_2 & b_2 & -c_2 & d_2 \\ -a_3 & b_3 & -c_3 & d_3 \end{vmatrix} = 0$$

and

$$\begin{vmatrix} 1 & 2\theta_2+\theta_1 & 2\theta_2\theta_1+\theta_2^2 & \theta_2^2\theta_1 \\ -a_1 & b_1 & -c_1 & d_1 \\ -a_2 & b_2 & -c_2 & d_2 \\ -a_3 & b_3 & -c_3 & d_3 \end{vmatrix} = 0,$$

or if, as before, $A = \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}$ &c. &c.,

these may be written

$$A + B(2\theta_1+\theta_2) + C(2\theta_1\theta_2+\theta_1^2) + D\theta_1^2\theta_2 = 0,$$

and

$$A + B(2\theta_2+\theta_1) + C(2\theta_2\theta_1+\theta_2^2) + D\theta_2^2\theta_1 = 0;$$

hence θ_1 and θ_2 are the roots of

$$(C^2-BD)\theta^2 + (BC-AD)\theta + B^2-AD = 0.$$

5262. (By S. Roberts, M.A.)—If s_0 is the number of integers less than N and prime to it, and s_1, s_2, s_3 are sums of those integers, of their squares and cubes respectively, prove that the roots of

$$s_0\alpha^3 + 3s_1\alpha^2 + 3s_2\alpha + s_3 = 0$$

are in arithmetical progression.

I. *Solution by* Ll. Wansbrough Jones; E. Rutter; *and others.*

We have [see Wolstenholme's *Book of Problems*, Ex. 231]

$$s_0 = N\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\ldots\ldots,$$

$$s_1 = \tfrac{1}{2}N^2\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\ldots\ldots,$$

$$s_2 = \tfrac{1}{3}N^3\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\ldots\ldots + \tfrac{1}{6}N(1-a)(1-b)\ldots\ldots,$$

$$s_3 = \tfrac{1}{4}N^4\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\ldots\ldots + \tfrac{1}{4}N^2(1-a)(1-b)\ldots\ldots$$

where a, b, ... are the different prime factors of N.

The necessary and sufficient condition that the roots of

$$s_0 a^3 + 3s_1 a^2 + 3s_2 a + s_3 = 0$$

may be in arithmetical progression is clearly that $-\frac{s_1}{s_0}$ or $-\frac{1}{2}N$ may be one of them, that is, that

$$s_0 \cdot \tfrac{1}{8}N^3 - 3s_1 \cdot \tfrac{1}{4}N^2 + 3s_2 \cdot \tfrac{1}{2}N - s_3 = 0,$$

which, on substituting the values given above, is seen to be an identity.

II. *Solution by the* PROPOSER.

Mr. Thacker gave, in the *Cambridge and Dublin Mathematical Journal* (Vol. V., p. 148), expressions for the sums of the powers of numbers less than and prime to a given integer N; viz., for a number being the product of, say, three powers of primes,

$$s_0 = N\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right),$$

$$s_1 = \tfrac{1}{2}N^2\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right),$$

$$s_2 = \tfrac{1}{3}N^3\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right) + \tfrac{1}{6}N(1-a)(1-b)(1-c),$$

$$s_3 = \tfrac{1}{4}N^4\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right) + \tfrac{1}{4}N(1-a)(1-b)(1-c),$$

a, b, c being the indices of the prime factors.

From these relations we get

$$6Ns_2 - 4s_3 = N^2 s_0, \quad N = \frac{2s_1}{s_0};$$

whence

$$s_0^2 s_3 + 2s_1^3 - 3s_0 s_1 s_2 = 0,$$

which is the condition that the stated equation may have roots in arithmetical progression. It is, in fact, the leading term of a covariant.

III. *Solution by* C. LEUDESDORF, M.A.

Suppose $N = a^p$, a being a prime number. Since all the integers less than N are prime to it except a, $2a$, $3a$ ... $\frac{N}{a}a$, we have

$$s_0 = N - \frac{N}{a} = N\left(1-\frac{1}{a}\right),$$

$$s_1 = (1+2+\ldots+N) - \left(a+2a+\ldots+\frac{N}{a}a\right)$$

$$= \tfrac{1}{2}N(N+1) - a\frac{N}{2a}\left(\frac{N}{a}+1\right) = \tfrac{1}{2}N^2\left(1-\frac{1}{a}\right),$$

$$s_2 = (1^2+2^2+\ldots+N^2)-\left(a^2+2^2a^2+\ldots+\frac{N^2}{a^2}a^2\right)$$

$$= \tfrac{1}{6}N(N+1)(2N+1)-\tfrac{1}{6}a^2\frac{N}{a}\left(\frac{N}{a}+1\right)\left(\frac{2N}{a}+1\right)$$

$$= \tfrac{1}{3}N^3\left(1-\frac{1}{a}\right)+\tfrac{1}{6}N(1-a),$$

$$s_3 = (1^3+2^3+\ldots+N^3)-\left(a^3+2^3a^3+\ldots+\frac{N^3}{a^3}a^3\right)$$

$$= \tfrac{1}{4}N^2(N+1)^2-a^3\frac{N^2\left(\frac{N}{a}+1\right)^2}{4a^2} = \tfrac{1}{4}N^4\left(1-\frac{1}{a}\right)+\tfrac{1}{4}N^2(1-a).$$

When $N = a^p b^q c^r \ldots$ $a, b, c \ldots$ being prime numbers, then, just as it is proved in Todhunter's *Algebra* that

$$s_0 = N\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\ldots,$$

so we may prove that

$$s_1 = \tfrac{1}{2}N^2\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\ldots,$$

$$s_2 = \tfrac{1}{3}N^3\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\ldots+\tfrac{1}{6}N(1-a)(1-b)\ldots,$$

$$s_3 = \tfrac{1}{4}N^4\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\ldots+\tfrac{1}{4}N^2(1-a)(1-b)\ldots,$$

and it is easily seen that the relation

$$2s_1^3-3s_0s_1s_2+s_0^2s_3 = 0$$

holds between these. But this is clearly the condition that the cubic given should have three roots in A. P., for it is the condition that the third part of the sum of the roots $\left(-\frac{s_1}{s_0}\right)$ should be a root of the cubic.

[The values of s_2 and s_3 are given in Professor Wolstenholme's *Book of Problems*, Ex. 231.]

5303. (By Sir James Cockle, F.R.S.)—Given

$$\frac{d^2z}{dx^2}-a^2y^{2b}\frac{d^2z}{dy^2} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1);$$

find formulæ of reduction.

I. *Solution by* ROBERT RAWSON.

This question is an unsolved example in BOOLE's *Differential Equations*, p. 459, where it is stated to be integrable in finite terms when $b = \frac{2i}{2i \pm 1}$, as shewn by LEGENDRE. To this may be added another case, viz., it is soluble also when $b = \frac{2(i \pm 1)}{2i \pm 1}$, i being any positive integer.

By the relation $z = yz_1$ .....(2),

the equation (1) is readily transformed to

$$\frac{d^2z_1}{dx^2} - a^2y^{2b}\frac{d^2z_1}{dy^2} - 2a^2y^{2b-1}\cdot\frac{dz_1}{dy} = 0 \text{.....} (3).$$

1. Change the independent variable y to t in each of the equations (1) and (3) respectively, by means of the relation $y = \beta t^n$ ..... (4).

Then $\frac{d^2z}{dx^2} - \frac{a^2\beta^{2b-2}t^{2bn-2n+2}}{n^2}\cdot\frac{d^2z}{dt^2} + \frac{a^2(n+1)\beta^{2b-2}t^{2bn-2n+1}}{n^2}\cdot\frac{dz}{dt} = 0 \text{...} (5)$,

and

$$\frac{d^2z}{dx^2}z_1 - \frac{a^2\beta^{2b-2}t^{2bn-2n+2}}{n^2}z_1\cdot\frac{d^2z}{dt^2} - \frac{a^2(n+1)\beta^{2b-2}t^{2bn-2n+1}}{n^2}\cdot\frac{dz_1}{dt} = 0 \text{...} (6).$$

If $a\beta^{b-1} = n$ and $bn - n + 1 = 0$ .....(7),

then equations (5) and (6) become simplified as follows:—

$$\frac{d^2z}{dx^2} - \frac{d^2z}{dt^2} - \frac{b}{(b-1)t}\cdot\frac{dz}{dt} = 0, \quad \frac{d^2z_1}{dx^2} - \frac{d^2z_1}{dt^2} - \frac{b-2}{(b-1)t}\cdot\frac{dz_1}{dt} = 0 \text{...} (8, 9).$$

Equation (1) depends, therefore, upon the solution of either equation (8) or (9). In fact, the elimination of t between (4) and the solution of either (8) or (9) is the solution of (1).

2. The reduction of (8) to a known soluble form is as follows:—

$$\frac{d^2v_{i-1}}{dx^2} - \frac{d^2v_{i-1}}{dt^2} - \frac{2(i-1)}{t}\cdot\frac{dv_{i-1}}{dt} = 0, \quad \frac{dv_{i-1}}{dt} = tv_i \text{.....} (10, 11);$$

then, i being any positive integer, $\frac{d^2v_i}{dx^2} - \frac{d^2v_i}{dt^2} - \frac{2i}{t}\frac{dv_i}{dt} = 0$.....(12);

when, however, i is taken $= 1$, it is well known that

$$v_0 = \phi(t+x) + \psi(t-x) \text{.....} (13).$$

From (12) and (10) the following is obvious:—

$v_1 = \frac{1}{t}\frac{d}{dt}(v_0)$ is the solution of $\frac{d^2v_1}{dx^2} - \frac{d^2v_1}{dt^2} - \frac{2}{t}\frac{dv_1}{dt} = 0$

$v_2 = \frac{1}{t}\frac{d}{dt}(v_1)$ " $\frac{d^2v_2}{dx^2} - \frac{d^2v_2}{dt^2} - \frac{2}{t}\frac{dv_2}{dt} = 0$

$\vdots$ $\vdots$

$v_i = \frac{1}{t}\frac{d}{dt}(v_{i-1})$ " $\frac{d^2v_i}{dx^2} - \frac{d^2v_i}{dt^2} - \frac{2i}{t}\frac{dv_i}{dt} = 0 \text{...} (14).$

If, then, the operation $\frac{1}{t}\frac{d}{dt}$ be performed i times upon v_0, the result is v_i.

If $$\frac{b}{b-1} = 2i, \quad \text{or } b = \frac{2i}{2i-1} \ldots\ldots\ldots\ldots\ldots\ldots (15);$$

then z in equation (8) will coincide with v_i in equation (14).

Again, put in

$$\frac{d^2v_{i-1}}{dx^2} - \frac{d^2v_{i-1}}{dt^2} + \frac{2(i-1)}{t}\frac{dv_{i-1}}{dt} = 0, \quad v_i = t^{2i}\frac{d}{dt}\left(\frac{v_{i-1}}{t^{2i-1}}\right) \ldots (16, 17);$$

then, $$\frac{d^2v_i}{dx^2} - \frac{d^2v_i}{dt^2} + \frac{2i}{t}\frac{dv_i}{dt} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (18);$$

from (17) and (18) it follows that v_0 is the same as in (13), and

$$v_1 = t^2\frac{d}{dt}\left(\frac{v_0}{t}\right) \text{ is the solution of } \frac{d^2v_1}{dx^2} - \frac{d^2v_1}{dt^2} + \frac{2}{t}\frac{dv_1}{dt} = 0$$

$$v_2 = t^4\frac{d}{dt}\left(\frac{v_1}{t^3}\right) \quad \text{,,} \quad \frac{d^2v_2}{dx^2} - \frac{d^2v_2}{dt^2} + \frac{4}{t}\frac{dv_2}{dt} = 0$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$v_i = t^{2i}\frac{d}{dt}\left(\frac{v_{i-1}}{t^{2i-1}}\right) \quad \text{,,} \quad \frac{d^2v_i}{dt^2} - \frac{d^2v_i}{dt^2} + \frac{2i}{t}\frac{dv_i}{dt} = 0 \ldots (19).$$

If $$\frac{b}{b-1} = -2i, \quad \text{or } b = \frac{2i}{2i+1} \ldots\ldots\ldots\ldots\ldots\ldots (20),$$

then z in equation (8) coincides with v_i in equation (19).

Hence (1) is soluble when $b = 2i$ upon $(2i \pm 1)$ as shewn by Legendre. By reference to equation (9) equation (1) is also soluble when $b-2$ upon $b-1$ is equal to $\pm 2i$, or $b = 2(i \pm 1)$ upon $2i \pm 1$.

3. By the conditions $n = -1$ and $\beta = 1$, equation (6) becomes

$$\frac{d^2z_1}{dx^2} - a^2t^{2(2-b)}\frac{d^2z_1}{dt^2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (21),$$

which is of the same form as is equation (1).

II. *Solution by the* Proposer.

1. Consider the equation

$$\frac{d^2z}{dx^2} - a^2y^{2b}\frac{d^2z}{dy^2} - a^2\beta y^{2b-1}\frac{dz}{dy} = 0 \ldots\ldots\ldots\ldots (2),$$

and make $y = t^n$. Then we have

$$y^{2b}\frac{d^2z}{dy^2} + \beta y^{2b-1}\frac{dz}{dy} \text{ will become } \frac{d^{2v}}{n^2}\left(\frac{d^2z}{dt^2} + \frac{\beta n - n + 1}{t}\frac{dz}{dt}\right)$$

where $v = bn - n + 1$. For $\frac{d^2z}{dy^2} + \frac{\beta}{y}\frac{dz}{dy}$ becomes

$$\left(\frac{dt}{dy}\right)^2\left\{\frac{d^2z}{dt^2} + \left(\frac{\beta}{y}\frac{dy}{dt} - \frac{dt}{dy}\frac{d^2y}{dt^2}\right)\frac{dz}{dt}\right\}.$$

Hence follows readily the result next preceding, and also the transformation of (2) into

$$\frac{d^2z}{dx^2} - \frac{a^2}{n^2} t^{2(bn-n+1)} \left(\frac{d^2z}{dt^2} + \frac{\beta n - n + 1}{t} \frac{dz}{dt} \right) = 0 \quad \ldots\ldots\ldots \quad (3).$$

2. Let, first, $\beta = 0$ and $n = -1$. Then (3) becomes

$$\frac{d^2z}{dx^2} - a^2 t^{2(2-b)} \left(\frac{d^2z}{dt^2} + \frac{2}{t} \frac{dz}{dt} \right) = 0 \quad \ldots\ldots\ldots \quad (4).$$

Substitute $\frac{u}{t}$ for z in (4) and multiply the result into u. Then (4) becomes

$$\frac{d^2z}{dx^2} - a^2 t^{2(2-b)} \frac{d^2z}{dt^2} = 0 \quad \ldots\ldots\ldots \quad (5).$$

In (5) replace t by y and compare with (1). It will be seen that we have changed b into $2-b$. Let $2-b = \phi(b)$, then, ϕ being a periodic function of the second order, we can proceed no further.

3. Let, secondly, $\beta = 2b$. Then (2) is a transformation of (1) obtained by differentiating (1) with respect to y and replacing $\frac{dz}{dy}$ by z in the result. Hence, if $2bn - n + 1 = 0$, then $n = \frac{1}{1-2b}$, and

$$(b-1)n + 1 = \frac{b-1}{1-2b} + 1 = \frac{b}{2b-1}.$$

Consequently (3) becomes

$$\frac{d^2z}{dx^2} - A^2 t^{\frac{b}{2b-1}} \cdot \frac{d^2z}{dt^2} = 0 \quad \ldots\ldots\ldots \quad (6),$$

where $A = a(1-2b)$. In (6) replace t by y and x by $\frac{x}{1-2b}$; reduce and compare the result with (1). It will be seen that we have changed b into $\frac{b}{2b-1}$. Let $\frac{b}{2b-1} = \chi(b)$, then, the period of χ being of the second order, we can proceed no further.

4. Let, thirdly, $\beta = 2b$, as in the last case. But now put $2b - n + 1 = 2$. Then $n = \frac{1}{2b-1}$ and $(b-1)n + 1 = \frac{b-1}{2b-1} + \frac{3b-2}{2b-1}$, and (3) becomes

$$\frac{d^2z}{dx^2} - B^2 t^{\frac{3b-2}{2b-1}} \left(\frac{d^2z}{dt^2} + \frac{2}{t} \frac{dz}{dt} \right) = 0 \quad \ldots\ldots\ldots \quad (7),$$

where $B = a(2b-1)$. In (7) replace x by $\frac{x}{2b-1}$, and t by y and z by $\frac{z}{y}$ and reduce. The result will be

$$\frac{d^2z}{dx^2} - a^2 y^{\frac{3b-2}{2b-1}} \frac{d^2z}{dy^2} = 0 \quad \ldots\ldots\ldots \quad (8),$$

and we have changed b into $\psi(b)$ where $\psi(b) = \frac{3b-2}{2b-1}$.

5. When we start from $b=0$, we obtain $\phi(0)=2$, $\phi(2)=0$; $\chi(0)=0$, $\chi(2)=\frac{2}{3}$, $\chi(\frac{2}{3})=2$; $\psi(0)=2$, $\psi(2)=\frac{4}{3}$, $\psi(\frac{4}{3})=\frac{6}{5}$, $\psi(\frac{6}{5})=\frac{8}{7}$, and so on.

For $$\psi\left(\frac{2m}{2m-1}\right)=\frac{6m-4m+2}{4m-2m+1}=\frac{2(m+1)}{2(m+1)-1} \quad \ldots\ldots\ldots (9),$$

whence $$\psi^m(0)=\frac{2m}{2m-1} \quad \ldots\ldots\ldots (10),$$

and, at all events by a convention of notation,

$$\psi^{-m}(0)=\frac{2m}{2m+1} \quad \ldots\ldots\ldots (11).$$

6. We also have $\psi\chi(2)=\psi(\frac{2}{3})=0$. Moreover

$$\phi\left(\frac{2m}{2m-1}\right)=2-\frac{2m}{2m-1}=\frac{2(m-1)}{2(m-1)+1},$$

a result which may be expressed by

$$\phi\psi^m(0)=\psi^{-(m-1)}(0) \quad \ldots\ldots\ldots (12).$$

Again, $$\chi\left(\frac{2m}{2m-1}\right)=\frac{2m}{2m+1}, \text{ or } \chi\psi^m(0)=\psi^{-m}(0) \ldots\ldots (13),$$

and, the period of ψ being infinite, we have an infinite number of results, the conclusion, in accordance with that of Legendre (see Boole, 2nd ed., p. 459), being that (1) is soluble when $b=\frac{2i}{2i\pm1}$, where i is an integer.

7. If we operate on $\psi^m(0)$ with ϕ and χ successively, we get

$$\phi\psi^m(0)=\psi^{-(m-1)}(0), \text{ and } \chi\psi^{-(m-1)}(0)=\psi^{m-1}(0),$$

so that $\phi\chi$ seems to be equivalent to ψ^{-1}.

5144. (By J. C. Malet, M.A.)—If we denote by $F(m, n, p)$ the determinant

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha^m & \beta^m & \gamma^m & \delta^m \\ \alpha^n & \beta^n & \gamma^n & \delta^n \\ \alpha^p & \beta^p & \gamma^p & \delta^p \end{vmatrix}$$

prove the formula $(\alpha+\beta+\gamma+\delta)F(m, n, p)$

$$=\alpha\beta\gamma\delta F(m-1, n-1, p-1)+F(m+1, n, p)+F(m, n+1, p)+F(m, n, p+1);$$

and generalise for a similar determinant of any order.

I. *Solution by* J. HAMMOND; R. E. RILEY, B.A.; *and others.*

Let $F(m, n, p, \ldots) = \begin{vmatrix} 1 & 1 & 1 & 1 & \ldots \\ \alpha^m & \beta^m & \gamma^m & \delta^m & \ldots \\ \alpha^n & \beta^n & \gamma^n & \delta^n & \ldots \\ \alpha^p & \beta^p & \gamma^p & \delta^p & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}$

and let (i, κ) denote the minor of the constituent in the i^{th} row and κ^{th} column. Then

$$\begin{aligned} F(m, n, p, \ldots) &= (1, 1) + (1, 2) + (1, 3) + \ldots \\ &= \alpha^m(2, 1) + \beta^m(2, 2) + \gamma^m(2, 3) + \ldots \\ &= \alpha^n(3, 1) + \beta^n(3, 2) + \gamma^n(3, 3) + \ldots \\ &= \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ &= (1, 1) + \alpha^m(2, 1) + \alpha^n(3, 1) + \ldots \\ &= (1, 2) + \beta^m(2, 2) + \beta^n(3, 2) + \ldots \\ &= \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \end{aligned}$$

Now $\alpha\beta\gamma\delta \ldots F(m-1, n-1, p-1, \ldots) = \alpha(1, 1) + \beta(1, 2) + \ldots,$

$F(m+1, n, p, \ldots) = \alpha^{m+1}(2, 1) + \beta^{m+1}(2, 2) + \ldots,$

$F(m, n+1, p, \ldots) = \alpha^{n+1}(3, 1) + \beta^{n+1}(3, 2) + \ldots,$

$F(m, n, p+1, \ldots) = \alpha^{p+1}(4, 1) + \beta^{p+1}(4, 2) + \ldots.$

Adding these, we obtain

$$\begin{aligned} &\alpha\{(1, 1) + \alpha^m(2, 1) + \alpha^n(3, 1) + \ldots\} \\ &\qquad + \beta\{(1, 2) + \beta^m(2, 2) + \beta^n(3, 2) + \ldots\} + \&c. \\ &= (\alpha + \beta + \gamma + \delta + \ldots) F(m, n, p, \ldots). \end{aligned}$$

II. *Solution by* HERBERT W. HARRIS; Professor EVANS, M.A.; *and others.*

Call the minors of $F(m, n, p)$, with regard to the *elements* in the first row, M_1, M_2, &c.; with regard to α^p, $\frac{dF}{d(\alpha^p)}$, and so on; then we have

$F(m, n, p) \Sigma(\alpha) =$

$$\begin{vmatrix} \alpha & \beta & 1 & 1 \\ \alpha^{m+1} & \beta^m & \gamma^m & \delta^m \\ \alpha^{n+1} & \beta^n & \gamma^n & \delta^n \\ \alpha^{p+1} & \beta^p & \gamma^p & \delta^p \end{vmatrix} + \begin{vmatrix} 1 & \beta & 1 & 1 \\ \alpha^m & \beta^{m+1} & \gamma^m & \delta^m \\ \alpha^n & \beta^{n+1} & \gamma^n & \delta^n \\ \alpha^p & \beta^{p+1} & \gamma^p & \delta^p \end{vmatrix} + \ldots + \ldots,$$

and similar determinants corresponding to γ, δ. Selecting suitable terms from these determinants, it is easy to see that their sum equals

$$\Sigma\alpha M_1 + \Sigma\alpha^{m+1}\frac{dF}{d(\alpha^m)} + \Sigma\alpha^{n+1}\frac{dF}{d(\alpha^n)} + \Sigma\alpha^{p+1}\frac{dF}{d(\alpha^p)},$$

and $M\Sigma\alpha M_1$ is seen on inspection to equal

$$\alpha\beta\gamma\delta F(m-1, n-1, p-1),$$

Also $$\Sigma a^{m+1}\frac{dF}{d(a^m)} = F(m+1, n, p),$$

and so on, whence the truth of the theorem is evident.

To generalize, call the determinant

$$\begin{vmatrix} 1 & 1 & 1 & \ldots\ldots & 1 \\ a_1^{m_1} & a_2^{m_1} & a_3^{m_1} & \ldots\ldots & a_n^{m_1} \\ a_1^{m_2} & a_2^{m_2} & a_3^{m_2} & \ldots\ldots & a_n^{m_2} \\ \ldots & \ldots & \ldots & \ldots\ldots & \ldots \\ a_1^{m_{n-1}} & a_2^{m_{n-1}} & a_3^{m_{n-1}} & \ldots\ldots & a_n^{m_{n-1}} \end{vmatrix}$$

$F(m_1 m_2 \ldots m_{n-1})$; then

$$F(m_1 m_2 \ldots m_{n-1})\,\Sigma a_1 = a_1 a_2 \ldots a_n F(m_1-1, m_2-1 \ldots m_{n-1}-1) + F(m_1+1 \ldots m_{n-1}) + F(m_1, m_2+1 \ldots m_{n-1}) + \text{etc.},$$

and the method of proof adopted above, being perfectly general, will evidently apply to both cases.

5223. (By J. J. Walker, M.A.)—Show that, for all systems of rectangular axes drawn through a given point in a given plane area, in its plane,

$$\left\{\iint (x^2-y^2)\,dx\,dy\right\}^2 + 4\left\{\iint xy\,dx\,dy\right\}^2,$$

taken over the whole of the area, is constant; and that for a triangle, the point being its centroid, this constant value is

$$(\tfrac{1}{18}\Delta)^2(a^4+b^4+c^4-b^2c^2-c^2a^2-a^2b^2).$$

I. *Solution by* Professor Lloyd Tanner, M.A.

This is a particular case of the following theorem; if

$$A = \iint (x^2+y^2)^n \left(x^m - \frac{\lfloor m}{\lfloor 2 \lfloor m-2}x^{m-2}y^2 + \frac{\lfloor m}{\lfloor 4 \lfloor m-4}x^{m-4}y^4 - \&c.\right) dx\,dy,$$

$$B = \iint (x^2+y^2)^n \left(\frac{\lfloor m}{\lfloor 1 \lfloor m-1}x^{m-1}y - \frac{\lfloor m}{\lfloor 3 \lfloor m-3}x^{m-3}y^3 + \&c.\right) dx\,dy,$$

(where m is a positive integer,) be taken over the whole of any given area;

then $A^2 + B^2$ is constant for all systems of rectangular axes having a common origin. In fact, changing to polar coordinates, we have

$$A = \iint r^{2n+m+1} \cos m\theta \, . \, dr\, d\theta, \quad B = \iint r^{2n+m+1} \sin m\theta \, . \, dr\, d\theta.$$

Now, if we take a new set of rectangular axes with the same origin, inclined at an angle α to the former set; we find A′, B′ for the new set by writing $\theta + \alpha$ for θ in A, B. Thus

$$A' = A \cos m\alpha - B \sin m\alpha, \quad B' = A \sin m\alpha + B \cos m\alpha,$$

therefore $A'^2 + B'^2 = A^2 + B^2$, or $A^2 + B^2$ is constant for all systems of rectangular axes through the given point.

Mr. WALKER's case corresponds to the values $n = 0$, $m = 2$.

II. *Solution by the* PROPOSER.

If the axes are turned through an angle θ, the new coordinates of any elements of the area are

$$x' = x \cos\theta - y \sin\theta, \quad y' = x \sin\theta + y \cos\theta,$$

whence

$$x'^2 - y'^2 = (x^2 - y^2) \cos 2\theta - 2xy \sin 2\theta,$$

$$2x'y' = (x^2 - y^2) \sin 2\theta + 2xy \cos 2\theta,$$

and, supposing $dx'\, dy' = dx\, dy$,

$$\left\{\iint (x'^2 - y'^2)\, dx'\, dy'\right\}^2 + 4\left\{\iint x'y'\, dx'\, dy'\right\}^2 = \left\{\iint (x^2 - y^2)\, dx\, dy\right\}^2 + 4\left\{\iint xy\, dx\, dy\right\}^2.$$

To calculate these integrals for a plane triangle ABC, the axes being drawn through the C of G, G parallel and perpendicular to BC, let AD, perpendicular on BC $= h$, and the lengths of the segments BD, BC be represented by m, n. Then over the whole area

$$\iint x^2\, dx\, dy = \frac{1}{81h^3} \int_{-\frac{2h}{3}}^{\frac{h}{3}} \left\{(ah + 3my)^3 + (ah + 3xy)^3\right\} dy$$

$$= \frac{h}{972}\left\{\frac{(2m+n)^4 - (m-n)}{m} + \frac{(2n+m)^4 - (n-m)^4}{n}\right\}$$

$$= \frac{\Delta}{72} \frac{3a^4 + (b^2 - c^2)^2}{a^2}.$$

Similarly may be calculated

$$\iint y^2\, dx\, dy = \frac{\Delta}{72} \frac{2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4}{a^2};$$

or this may be deduced from the expression of the moment of inertia. Also

$$\iint xy\, dx\, dy = \frac{\Delta}{36} h\,(m - n),$$

or

$$4\left\{\iint xy\, dx\, dy\right\}^2 = \left(\frac{\Delta}{72}\right)^2 \frac{4\,(2b^2c^2 + \ldots - c^4)\,(b^2 - c^2)^2}{a^4};$$

therefore $$\left\{\iint (x^2-y^2)\,dx\,dy\right\}^2+4\left\{\iint xy\,dx\,dy\right\}^2 = \left(\frac{\Delta}{18}\right)^2(a^4+b^4+c^4-b^2c^2-c^2a^2-a^2b^2).$$

5273. (By Professor CROFTON, F.R.S.)—If a point is taken at random in a triangle ABC, and three other points are taken at random; show that (1) the probability that all three fall on one of the triangles BXC, CXA, AXB is $\frac{3}{10}$; and (2) the probability that they fall, one on each triangle, is $\frac{1}{10}$.

I. *Solution by* E. B. ELLIOTT, M.A.

If x, y, z are the ratios of the areas BXC, CXA, AXB to ABC, all positive values of x, y, z which satisfy $x+y+z=1$ are equally likely.

1. Now x^3 is the chance that all three random points lie in BXC, corresponding to a definite position of X; and y^3 is the chance that all lie in CXA, and z^3 in AXB; thus the chance that all lie in the same one of the triangles is $x^3+y^3+z^3$.

Thus, what is required is the mean value of $x^3+y^3+z^3$, that is to say, of $3x^3$, as X takes all possible positions. It is then

$$\frac{3\iint x^3\,dx\,dy}{\iint dx\,dy}\ (\text{integrations over } x+y<1) = \frac{3\cdot\frac{1}{4.5}}{\frac{1}{2}} = \frac{3}{10}.$$

2. The chance that all lie in different triangles is, for a definite position of X, $6xyz$, since three things may be permuted in six ways. Thus the chance required is the mean of $6xyz$, and is therefore (limits as before)

$$\frac{6\iint xy\,(1-x-y)\,dx\,dy}{\iint dx\,dy} = \frac{6\left\{\frac{1}{2.3.4}-\frac{1}{3.4.5}-\frac{1}{3.4.5}\right\}}{\frac{1}{2}} = \frac{1}{10}.$$

II. *Solution by* E. B. SEITZ; ELIZABETH BLACKWOOD; *and others.*

Let p_1x, p_2y, p_3z be the perpendiculars from X on BC, CA, AB, p_1, p_2, p_3 being the perpendiculars from A, B, C on BC, CA, AB. Then we have

$$ap_1x+bp_2y+cp_3z = ap_1 = bp_2 = cp_3\,2\Delta \quad \ldots\ldots\ldots\ldots\ (a);$$

whence $$z = 1-x-y \quad \ldots\ldots\ldots\ldots\ (b);$$

and for a given position of X the probability in (1) is $x^3+y^3+z^3$, and that in (2) is $6xyz$.

Hence, substituting the value z from (b), we have for (1)

$$p = \frac{1}{\Delta}\int_0^1\int_0^{1-x} .(x^3+y^3+z^3)\,p_1\,dx\,p_2\,\mathrm{cosec}\,\mathrm{C}\,dy$$

$$= 2\int_0^1\int_0^{1-x}[1-3x+3x^2-3y(1-x)^2+3y^2(1-x)]\,dx\,dy$$

$$= \int_0^1 (1-4x+6x^2-2x^3-x^4)\,dx = \tfrac{3}{10}.$$

For (2) we have

$$p = \frac{1}{\Delta}\int_0^1\int_0^{1-x} 6xyz\,p_1\,dx\,p_2\,\mathrm{cosec}\,\mathrm{C}\,dy$$

$$= 12\int_0^1\int_0^{1-x}(xy-x^2y-xy^2)\,dx\,dy$$

$$= 2\int_0^1 (1-x)^2x\,dx = \tfrac{1}{10}.$$

5164. (By J. J. Walker, M.A.)—If

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2lx+2my+2nz+d = 0$$

be the equation to a pair of planes, prove that

1. The direction-cosines of the angles α, β, γ which the line of intersection of the planes makes with the (rectangular) axes are given by

$$\sec\alpha : \sec\beta : \sec\gamma = gh-af : hf-bg : fg-ch.$$

2. The product of the perpendiculars let fall from the point (x', y', z') on the planes is equal to

$$\frac{ax'^2+\ldots+2fy'z'+\ldots+2lx'+\ldots+d}{\left\{(a+b+c)^2+4(f^2-bc)+4(g^2-ca)+4(h^2-ab)\right\}^{\frac{1}{2}}}.$$

I. *Solution by* G. S. Carr.

1. Transforming to the centre so as to remove the terms in x, y, z, the first six terms will not be altered, and the direction-cosines of the intersection of the planes will also remain unchanged.

The equations of the line of intersection are shown in Frost and Wolstenholme's *Solid Geometry*, Art. 222, to be

$$x(gh-af) = y(hf-bg) = z(fg-ch).$$

Hence the reciprocals of the coefficients of x, y, z are the direction-cosine-required.

2. To find the product of perpendiculars from x', y', z' on the two planes. Let the given equation of the planes be equivalent to

$$(Ax+By+Cz+D)(A'x+B'y+C'z+D') = 0.$$

Then the product of perpendiculars will be

$$\frac{(Ax'+By'+Cz'+D)(A'x'+B'y'+C'z'+D')}{\{(A^2+B^2+C^2)(A'^2+B'^2+C'^2)\}^{\frac{1}{2}}}.$$

Multiplying out the last equation, and equating coefficients with the original one, we have

$$AA' = a, \quad A'B+AB' = 2h, \quad \&c.$$

Multiplying together the factors of the denominator in the product, we obtain $AA'^2+(A^2B^2+A^2B'^2)+\&c.$ symmetrically

$$= AA'^2+(A'B+AB')^2-2AA'BB'+\&c. = a^2+4h^2-2ab+\&c.$$
$$= a^2+2ab+4(h^2-ab)+b^2+2bc+4(f^2-bc)+c^2+2ca+4(g^2-ca)$$
$$= (a+b+c)^2+4(f^2-bc)+4(g^2-ca)+4(h^2-ab),$$

the form of denominator required.

II. *Solution by* R. Tucker, M.A.; Professor Evans, M.A.; *and others.*

Let the planes be $Ax+By+Cz+D = 0$ ..... (1),

and $A'x+B'y+C'z+D' = 0$ ..... (2),

then we have $a = AA', \quad 2f = BC'+B'C,$

$b = BB', \quad 2g = AC'+A'C,$

$c = CC', \quad 2h = AB'+A'B.$

1. If α, β, γ be the direction-cosines of (1), (2), then (Salmon, p. 26)

$$\cos\alpha : \cos\beta : \cos\gamma = BC'-B'C : CA'-C'A : AB'-A'B$$
$$= (gh-af)^{-1} : (hf-bg)^{-1} : (fg-ch)^{-1}.$$

2. Product of perpendiculars from (x', y', z') is (Salmon, p. 21)

$$= \frac{(Ax'+By'+Cz'+D)(A'x'+B'y'+C'z'+D')}{\{(A^2+B^2+C^2)(A'^2+B'^2+C'^2)\}^{\frac{1}{2}}}$$

Now (denominator)^2

$$= (AA'+BB'+CC')^2+(AB'-A'B)^2+\ldots+\ldots$$
$$= (a+b+c)^2+4(h^2-ab)+\ldots+\ldots$$

therefore, &c.

5386. (By S. A. Renshaw.)—Let ABCD be a quadrilateral inscribed in a conic of focus F, and BCD any one of the four triangles of which it is composed; and let BC, CD, DB be produced to meet the directrix in P_2, P_3, P_6; and FP_2, FP_3, FP_6 be joined, and FQ_2, FQ_3, FQ_6 be drawn perpendicular to them, meeting the directrix in Q_2, Q_3, Q_6; then prove that (1),

if Q_2A, Q_3A, Q_6A be joined meeting BC, CD, and DB in a_1, a_2, a_3, these three points a_1, a_2, a_3 will be in a straight line; (2) if each of the remaining triangles CDA, DAB, CDB be similarly treated in conjunction with the vertices B, C, D (*i.e.*, CDA with B, &c.), there will be obtained similarly three other sets of three points each, viz., (b_1, b_2, b_3), (c_1, c_2, c_3), (d_1, d_2, d_3), each set lying in a straight line, and the four lines containing the four sets will all pass through the same point O.

Solution by C. TAYLOR, M.A.

Making use of the method described in the *Quarterly Journal of Pure and Applied Mathematics*, Volume xiv., pp. 30—34, and retaining the notation of the original figure, let the conic be transformed into a circle about F as centre. Then, since (1) points on the directrix go to *infinity*, and (2) angles subtended at F are *unaltered* by the transformation, therefore AQ_2 is (parallel to FQ_2, and) perpendicular to (FP_2 and) BC. That is to say, Aa_1 is the perpendicular let fall upon a side of BCD from a point A on its circumscribing circle. And, by known properties of the circle, the feet of such perpendiculars are collinear, three and three, &c. &c. &c.

5314. (By ALEXANDER J. ELLIS, F.R.S.)—Given $f(ax, y) = af(x, y)$ and $f(x, by) = bf(x, y)$, to find all the possible forms of $f(x, y)$. It is usually assumed that $f(x, y) = cxy$, where c is any constant, and this form evidently satisfies the conditions. But if $a = b$, then $f(x, y) = c \div xy$ also satisfies them; and generally, if $a^m = a$, $b^n = b$, that is, if a is any one of the m roots of a, and b any one of the n roots of b, then $f(x, y) = x^m y^n$ also satisfies the conditions. The question becomes important in such problems as this:—The work done in carrying a load over a distance is multiplied when either the load or the distance is multiplied; what is the relation between work done, load, and distance?

Solution by Professor LLOYD TANNER, M.A.

If we put $$f(x, y) = xy f_1(x, y),$$

the given functional equations become

$$f_1(x, y) = f_1(ax, y) = f_1(x, by).$$

Hence f_1 is any function which is not altered in value by changing x, y into ax, by. The general form of f_1 is

$$\sum_{m=0}^{m=\infty} \sum_{n=0}^{n=\infty} C_{m,n} \cos\left(2m\pi \frac{\log x}{\log a} + A_m\right) \cos\left(2n\pi \frac{\log y}{\log b} + B_n\right),$$

where the A, B, C's are constants. The solution quoted by Mr. Ellis, viz.,

$$f(x, y) = cxy \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

ought to be accompanied with the remark that c is an abbreviation for the above trigonometrical series.

The statement that, if $a = b$, then $f(x, y) = c \div xy$ satisfies the equations, appears to be erroneous. On substituting this form, we get the equations $a = \pm 1$, $b = \pm 1$. Taking the upper signs, the equations become identities; taking the lower signs, f may be any odd function of x, y.

The illustrative problem, relating to work, is essentially different from the one just discussed. In this we are given not only the two functional equations, but also that they are satisfied for all values of a, b. Here, then, $f_1(x, y)$ is not changed when x is changed to ax, whatever a may be; *i. e.*, it is not affected by *any* change in the value of x, or it is constant with respect to x. Similarly, it is constant with respect to y. In this case (1) is the general solution, c being now a constant.

Note on Professor Tanner's Solution of Question 5314.

By Alexander J. Ellis, F.R.S.

The second sentence in the Question should have begun—"But if $a^{-1} = a$ and $b^{-1} = b$," &c., being a particular case of "generally, if $a^m = a$, $b^n = b$," which follows. Of course the third sentence was connected with the first; and the intention of asking the question was to get the general form of c. But I am not quite sure that Prof. Tanner has given the *general* solution, for his formula evidently holds only when $\log ax = \log a + \log x$, which is not the case, except with certain limitations, when a and x are negative or imaginary. Also I am not clear that the meaning of my third sentence implied that $f_1(x, y)$ is "not affected by any change in the value of x" or y. I meant certainly only that it should remain constant if a, b, x, y were positive and possible; and this is precisely the case contemplated in the above solution. If so, c is not necessarily constant in this case.

Note on the Solution of Question 5314.

By Professor Lloyd Tanner, M.A.

Mr. Ellis's objections to my solution of Quest. 5314 may be met thus:—

1. Let us take $\log x$ to mean $\int_1^x \frac{dt}{t}$; and suppose that t is situated on a Riemann's screw surface. Then $\log x$ becomes a single-valued function of x, and satisfies the equation $\log ax = \log a + \log x$, where a, x may be any complex imaginary quantities. Unless this, or some equivalent restriction be assumed, the value of f will not always satisfy the given equations.

2. We cannot, of course, say that all solutions of the given equation are included in the general solution found. But we can say that any form of f which can be expressed in a trigonometrical series is included; and this class comprises, if I mistake not, all continuous functions and most known discontinuous functions. For instance, Mr. Ellis's particular solutions are included. Say we have $f(x, y) = x^{r+1} y^{s+1}$, so that, if $i = (-1)^{\frac{1}{2}}$,

$$f_1(x, y) = x^r y^s = e^{r \log x} \cdot e^{s \log y}$$

$$= \left\{ \cos \frac{r \log x}{i} + i \sin \frac{r \log x}{i} \right\} \left\{ \cos \frac{s \log y}{i} + i \sin \frac{s \log y}{i} \right\} \quad \ldots\ldots(2).$$

Now, if in the general form of f_1 we write

$$A \cos \xi + B \sin \xi \text{ for } A' \cos (\xi + B'),$$

it clearly includes (2) provided only that

$$\frac{r}{i} = \frac{2m\pi}{\log a}; \quad \frac{s}{i} = \frac{2n\pi}{\log b}.$$

From the first of these $r \log a = 2m\pi i$, therefore $a^r = 1$; and similarly $b^s = 1$. These are equivalent to Mr. Ellis's conditions.

3. If it be admitted that when x is real and positive $f_1(x)$ is constant, it will follow that $f_1(x)$ is constant for all values, real or imaginary, of x. This is a particular case of the theorem, "Einer Function $w = u + vi$ von $z (= x + yi)$ kann nicht längs einer Linie constant sein wenn sie nicht überall constant ist." (Riemann, Inauguraldissertation, Art. 15). It is assumed that f_1 or w is continuous save at isolated points; otherwise the problem is indeterminate.

4200. (By the Editor.)—A variable square moves with its centre on the circumference of a given circle, and has one of its diagonals always parallel to a given diameter of the circle, and equal to twice the distance of the centre of the square from one end of the given diameter; show that the corners of the square move on the curves

$$r = 2a(\cos\theta \pm \cos 3\theta) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1, 2),$$

the given diameter ($=2a$) being the polar axis, and its end the origin. Trace these curves, showing that (1), with the upper sign, is the locus of the corners at the ends of the diagonal parallel to the given diameter; that the Cartesian equation of the curve is

$$(x^2+y^2)^2 = 4ax(x^2-y^2), \text{ or } \left\{y^2 + x(2a+x)\right\}^2 = 4ax^2(a+2x) \ldots (3);$$

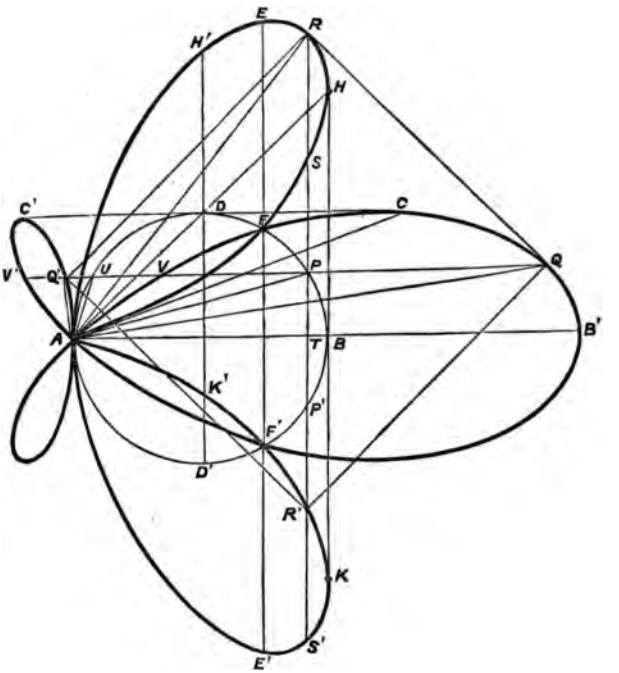
that the origin is a triple point where two branches form a crunode, cutting the axis at angles of 45°, and that the third branch, near the origin, takes the form of the parabola $y^2 = -4ax$; that the curve consists of three loops, one, of length $4a$ and breadth $2a$, to the right of the origin, and two equal loops extending to a distance $\frac{1}{2}a$ to the left of the origin:—also that (2), with the lower sign, is the locus of the ends of the diagonal perpendicular to the fixed diameter; that the Cartesian equation of the curve is

$$(x^2+y^2)^2 = 8axy^2, \text{ or } \left\{y^2 - 2x(4a-x)\right\}^2 = 8ax^2(2a-x) \ldots\ldots (4);$$

that the origin is a triple point where two of the branches form a ceratoid cusp, which, near the origin, takes the form of the semi-cubical parabola $x^3 = 8ay^2$, and the third branch, near the origin, takes the form of the parabola $y^2 = 8ax$; that the curve consists of two equal loops which extend to a distance of $2a$ to the right of the origin, and cut the large-loop branches of (1) on the circle, at an angular ordinate of 30°; that the chord of (1) parallel to the given diameter, and the chord of (2) perpendicular to the given diameter, are each double the corresponding chord of the circle; and that the whole area of each curve is double the area of the fixed circle.

Solution by the PROPOSER.

Let P be the centre of the square QRQ'R', AB the given diameter of the circle APB, and ϕ ($= \angle PAB$) the angular ordinate of the centre P of



the square; then, to find the polar equation of the locus of Q, we have

$$AP = r, \quad \angle QAB = PQA = PAQ = \theta, \quad \phi = PAB = 2\theta;$$

therefore $r = 2AP \cos PAQ = 4a \cos\theta \cos 2\theta = 2a(\cos\theta + \cos 3\theta)$ ...(5).

It is clear that the locus of the other end Q' of the diagonal QQ' must also be the curve (1); but this locus may be investigated with equal ease from the figure; for, in this case,

$$\angle Q'AB = \theta, \quad \angle QAQ' = \tfrac{1}{2}\pi, \quad \angle PAQ' = PQ'A = \pi - \theta, \quad \phi = PAB = 2\theta - \pi;$$

therefore $r = 2AP \cos PAQ' = 4a \cos 2\theta \cos\theta = 2a(\cos\theta + \cos 3\theta)$ ... (6).

For the locus of R (and similarly of R') we have

$$\angle RAB = \theta, \quad \angle PRA = PAR = \tfrac{1}{2}\pi - \theta, \quad \phi = PAB = 2\theta - \tfrac{1}{2}\pi;$$

therefore $r = 2AP \cos PRA = 4a \sin 2\theta \sin\theta = 2a(\cos\theta - \cos 3\theta)$ .... (7).

For the Cartesian equation of the curves, we have

$$(x^2+y^2)^2 = r^4 = 4ar^3 \cos\theta \cos 2\theta, \quad \text{or} \quad 4ar^3 \sin\theta \sin 2\theta,$$

which reduce to the forms given in the question.

Using Newton's process in the form given by Salmon (*Higher Plane Curves*, 2nd Ed., Art. 56) in order to find the form of the curves in the neighbourhood of the origin, let us substitute, in the equations of these curves, $y = \lambda x^n$; then equation (3) becomes

$$x^4 - 4ax^3 + 2\lambda^2 x^{2n+2} + 4a\lambda^2 x^{2n+1} + \lambda^4 x^{4n} = 0 \quad \ldots\ldots\ldots\ldots(8).$$

If, in (8), $n=1$, then $\quad (\lambda^2+1)^2 x^4 + 4a(\lambda^2-1)x^3 = 0 \quad \ldots\ldots\ldots\ldots(9).$

In (9) the coefficient of x^3 will vanish when $\lambda^2 = 1$; thus, $y = \pm x$ are the tangents at the origin to the two branches FA, F′AV′ of the branches of the locus of Q, which, therefore, cross the axis at angles of 45°; as might, in fact, have been at once seen from the factor x^2-y^2 in the equation of the curve.

If, in (8), $n=\frac{1}{2}$, then $\quad x^4 + 2(\lambda^2-2a)x^3 + \lambda^2(\lambda^2+4a)x^2 = 0 \quad \ldots\ldots\ldots(10).$

In (10) the coefficient of x^2 will vanish when $\lambda^2 = -4a$; thus the other two branches C′A of the locus of Q to the left of A take, near the origin, the form of the parabola $y^2 = -4ax$.

Substituting, in (4), $y = \lambda x^n$, the equation becomes

$$x^4 + 2\lambda^2 x^{2n+2} - 8a\lambda^2 x^{2n+1} + \lambda^4 x^{4n} = 0 \quad \ldots\ldots\ldots\ldots(11).$$

If, in (11), $n=\frac{1}{2}$, then $\quad x^4 + 2\lambda^2 x^3 + \lambda^2(\lambda^2-8a)x^2 = 0 \quad \ldots\ldots\ldots\ldots(12),$

whence, as above, $\lambda^2 = 8a$ shows us that, near A, the branches REAE′ of the locus of R take the form of the parabola $y^2 = 8ax$.

If, in (11), $n=\frac{3}{2}$, then $\quad \lambda^4 x^6 + 2\lambda^2 x^5 + (1-8a\lambda^2)x^4 = 0 \quad \ldots\ldots\ldots\ldots(13),$

whence, as before, $8a\lambda^2 = 1$ shows us that, near A, the two cusped-branches RFAF′R′ of the locus of R take the form of the semi-cubical parabola $x^3 = 8ay^2$.

For the chord-property, we have, in the figure,

$$\text{TR} = \text{PA} + \text{PT}, \quad \text{and} \quad \text{TS} = \text{TR}' = \text{PA} - \text{PT};$$

therefore $\quad \text{RS} = \text{R}'\text{S}' = 2\text{PT}, \quad \text{and} \quad \text{RS} + \text{R}'\text{S}' = 2\text{PP}' \quad \ldots\ldots\ldots\ldots(14).$

Again, $\quad \text{QV} = \text{PA} + \text{PU} - \text{UA}, \quad \text{and} \quad \text{Q}'\text{V}' = \text{PU} + \text{UA} - \text{PA};$

therefore $\quad \text{QV} + \text{Q}'\text{V}' = 2\text{PU} \quad \ldots\ldots\ldots\ldots(15).$

Since RS = 2PT, &c., it follows, from (14) and (15), that the whole area of each locus is double the area of the circle,—a result which may be obtained also by the aid of the Integral Calculus.

The curves in the Question may be readily traced. As P moves over the first quadrant of the circle, from B to D, where $\angle \text{DAB} = 45°$, the corners Q, R, Q′, R′ of the square trace the arcs B′C, HH′, AC′, KF′K′, where BH = BK = BA, and DC = DC′ = DH′ = DK′ = DA; as P moves over the second quadrant, from D to A, the arcs CFA, H′A, C′V′A, K′A are traced out; and as P moves over the third and fourth quadrants AD′, D′B′ of the circle, the corners of the square trace out the other halves of the curves, which are symmetrical with those already described.

Expressing the coordinates of a point on the curves as functions of θ, we have in the loci (1) and (2) of Q and R respectively

$$x = 2a\cos 2\theta(1+\cos 2\theta) = a(1+2\cos 2\theta+\cos 4\theta), \quad y = a\sin 4\theta \ldots(16);$$

$$x = a(1-\cos 4\theta), \quad y = 2a\sin 2\theta(1-\cos 2\theta) = a(2\sin 2\theta - \sin 4\theta) \ldots(17).$$

We thus see that, in (17), y is a *maximum* ($=a$) at C, when P is at D and $\phi = 2\theta = 45°$; and that x is a *maximum* ($=4a$) at B′, when P is at B and $\theta = 0$, and a *minimum* ($= -\frac{1}{4}a$) a little above V′, where $\theta = 120°$ and $\phi = 60°$. The latter is also clear from the second form of equation (3), wherein x cannot be $< -\frac{1}{4}a$, or the right-hand side of the equation would be *negative*.

In (18), we see likewise that x is a *maximum* ($=2a$) at H, when P is at B, $\phi = \theta$, and $\theta = 45°$; and that y is a *maximum* ($= \frac{3}{4}\sqrt{3} \cdot a$) at E, when $\theta = 60°$, $\phi = 30°$, and P is at F. The former appears also from the second form of equation (4), wherein x cannot evidently exceed $2a$.

The curves belong to the species $r = a \cos m\theta + b \cos n\theta$ .....(18), one of which has been drawn and discussed by the EDITOR, on pp. 59—62 of Vol. XIX. of the *Reprint*, as the pedal of the four-cusped hypocycloid.

5140. (By R. W. GENESE, M.A.)—Construct geometrically the chords which cut an ellipse at given angles α, β; and shew there are in general eight solutions.

Solution by Professor TOWNSEND, F.R.S.

The three angles α, β, γ of the triangle PQR determined by the required chord PQ with the tangents PR and QR at its extremities being all given, so therefore is the angle δ at which PQ is bisected at S by the diameter OR passing through R, which latter point lies consequently on one of the four diameters whose conjugates make with them the angle δ, and which may be determined by several well known constructions; any diameter of the tetrad being then taken, the two positions of R upon it may be determined immediately from the consideration that the two corresponding tangents PR and QR make known angles with it, and are therefore given in direction. There are therefore in general eight different solutions, which reduce to four in the two extreme cases of the equi-conjugate diameters, and of the axes, and to two when α and β are the angles at which those lines intersect the curve.

5363. (By Professor LLOYD TANNER, M.A.)—Solve the equation

$$\frac{d^2z}{dxdy} - \frac{1}{y}\frac{dz}{dx} + ue^{-z}\frac{dz}{dy} + \left(\frac{u}{y} - \frac{du}{dy}\right)e^{-z} = 0;$$

where u is any function of x, y.

I. *Solution by* L. H. ROSENTHAL.

Adopting the usual notation for partial differential coefficients, the equa-

tion may be written in the form

$$\frac{d}{dy} \cdot \{p - ue^{-z}\} - \frac{1}{y}\{p - ue^{-z}\} = 0.$$

Integrating with regard to y, we have

$$\frac{p - ue^{-z}}{y} = \phi'(x), \quad \text{or, if } e^z = t, \quad \frac{dt}{dx} - y\phi(x) \cdot t = u,$$

therefore, integrating with regard to x alone, the required solution is

$$z = y\phi(x) + \log\left\{\psi(y) + \int e^{-y\phi(x)} u dx\right\},$$

y being regarded as a constant in performing the integration on the right hand side of this equation.

II. *Solution by* ROBERT RAWSON.

This differential equation may be written as follows:—

$$e^z\left\{\frac{d^2z}{dxdy} - \frac{1}{y}\frac{dz}{dx}\right\} + u\frac{dz}{dy} + \frac{u}{y} - \frac{du}{dy} = 0 \ldots\ldots\ldots\ldots (1).$$

The form of (1) readily suggests the general form of its primitive, which may be denoted by the equation $e^z = R$.....(2), where R is an unknown function of x, y, to be determined so as to satisfy (1). Differentiate (2) as follows:—

$$e^z\frac{dz}{dx} = \frac{dR}{dx}, \quad \text{and} \quad e^z\frac{dz}{dy} = \frac{dR}{dy} \ldots\ldots\ldots\ldots (3).$$

Then
$$e^z\frac{d^2z}{dxdy} + e^z\frac{dz}{dx}\frac{dz}{dy} = \frac{d^2R}{dxdy},$$

or
$$e^z\frac{d^2z}{dxdy} + \frac{dR}{dx}\frac{dz}{dy} = \frac{d^2R}{dxdy}.$$

But
$$e^z \cdot v\frac{dz}{dx} = v\frac{dR}{dx},$$

where v is any function whatever;

therefore
$$e^z\left\{\frac{d^2z}{dxdy} - v\frac{dz}{dx}\right\} + \frac{dR}{dx}\frac{dz}{dy} + v\frac{dR}{dx} - \frac{d^2R}{dxdy} = 0 \ldots\ldots (4).$$

Now, the partial differential equation (4) has, evidently, for its solution the equation (2), because equation (2) and its derivatives will satisfy it.

The functions (v) and (R) can be determined so that equation (4) will coincide with (1) as follows.

Put $\quad v = \frac{1}{y} \quad \text{and} \quad \frac{dR}{dx} = u, \quad \therefore \frac{d^2R}{dxdy} = \frac{du}{dy}.$

Substitute these values in (4) and it becomes

$$e^z\left\{\frac{d^2z}{dxdy} - \frac{1}{y}\frac{dz}{dx}\right\} + u\frac{dz}{dy} + \frac{u}{y} - \frac{du}{dy} = 0,$$

whose primitive, or solution, is $e^z = \int u dx + y_1 \ldots\ldots\ldots\ldots\ldots (5),$

where y_1 is a function of y only.

Equation (5) is a particular solution of eq. 4; a general integral may be obtained as follows:—

Each of eqs. (3) is a first integral of eq. 4.—The following first integral of a more general differential equation than (4) is suggested by the fact that a factor and an arbitrary function of x are lost by the process of differentiation:

$$w\left\{\frac{dz}{dx} + \mathrm{R}e^{nz}\right\} = x_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6),$$

where w, R are functions of x, y, and x_1 a function of x.

Differentiate (6) with respect to y, then we have

$$\frac{d^2z}{dx\,dy} + \frac{dw}{w\,dy}\frac{dz}{dx} + e^{nz}\left\{n\mathrm{R}\frac{dz}{dy} + \mathrm{R}\frac{dw}{w\,dy} + \frac{d\mathrm{R}}{dy}\right\} = 0 \ldots\ldots (7).$$

Now (6) is the first integral of (7), and its solution is readily obtained to be as follows:—

$$e^{n\int\frac{x_1}{w}dx - nz} = n\int e^{n\int\frac{x_1}{w}dx}.\mathrm{R}\,dx + y_1 \ldots\ldots\ldots\ldots\ldots (8).$$

where y_1 is a function of y only.

III. *Solution by the* Proposer.

Mr. Rawson has kindly sent some equations including this as a particular case. Of these I take

$$\frac{d^2z}{dx\,dy} - \frac{1}{v}\cdot\frac{dv}{dy}\cdot\frac{dz}{dx} + \left(u\frac{dz}{dy} + \frac{u}{v}\cdot\frac{dv}{dy} - \frac{du}{dy}\right)e^{-z} = 0,$$

where u, v are any functions of x, y. This reduces to the proposed equation on putting $v = y$.

Dividing throughout by v, the expression on the left becomes an exact differential, and on integrating with respect to y, we get a first integral

$$\frac{1}{v}\left(\frac{dz}{dx} - ue^{-z}\right) = \phi(x),$$

whence

$$e^z = e^{\int\lambda\phi(x)\,dx}\left\{\int ue^{-\int\lambda\phi(x)\,dx}dx + \psi(y)\right\}.$$

2031. (By the Editor.)—Find the least positive value of x that will make the expression $927x^2 - 1236x + 413$ a rational square.

Solution by Dr. Hart.

Adopting the equality, as given in the Editor's Solution (*Reprint*, Vol. XX., p. 67), we must have $103(3x-2)^2 + 1 = \square$.

Put

$$103(3x-2)^2 + 1 = \left\{1 - \frac{m}{n}(3x-2)\right\}^2,$$

then $x = \frac{2m^2+2mn-206n^2}{3m^2-309n^2}$; whence, if $u = 0$, we have $x = \frac{2}{3}$.

This is the least value of x; and an infinite number of answers can be found by varying the values of m, n in the above general value of x. In my former solution of this Question (*Reprint*, Vol. XX., p. 66), it has been proved that integral answers to this problem are impossible.

[Dr. HART'S value $\frac{2}{3}$, here obtained for x, is equivalent to taking $p=1$, $q=0$ as a solution of the equation $p^2-103q^2 = 1$, into which the EDITOR has (in Vol. XX., p. 67) transformed the given question.]

4229. (By The EDITOR.)—Find rational plane triangles, whose sides are integers, such that two of the sides shall differ by unity; and show that 9, 10, 17; 13, 14, 15; 11, 13, 20; 17, 25, 26 are sides of four such triangles.

Solution by Dr. HART.

Let x, y, $y+1$ be the sides of the triangle, then we must have

$$\Delta = \frac{2y+x+1}{2} \cdot \frac{2y-x+1}{2} \cdot \frac{x+1}{2} \cdot \frac{x-1}{2} = \frac{(2y+1)^2-x^2}{4} \cdot \frac{x^2-1}{4} = \square,$$

which is so, if the numerator of each factor be taken $= 2\square$.

First, let $x^2-1 = 2m^2$, then $x^2 = 2m^2+1 = \square = \left(\frac{h}{q}m-1\right)^2$, suppose;

whence we have $m = \frac{2pq}{p^2-q^2}$, whence $x = \frac{p^2+2q^2}{p^2-2q^2}$ ..... (1).

Next, let $(2y+1)^2-x^2 = 2n^2$,

then $(2y+1)^2 = 2n^2+x^2 = \square = \left(\frac{r}{s}n+x\right)^2$, suppose;

whence we find $n = \frac{2rsx}{r^2-2s^2}$, therefore $2y+1 = \frac{(r^2+2s^2)x}{r^2-2s^2}$;

$\therefore y = \frac{(r^2+2s^2)x-(r^2-2s^2)}{2(r^2-2s^2)}$, and $y+1 = \frac{(r^2+2s^2)x+(r^2-2s^2)}{2(r^2-2s^2)}$ ...(2, 3).

Therefore (1), (2), (3) are general expressions for the sides of a scalene triangle; and, by substitution, the area is

$$\left(\frac{(2y+1)^2-x^2}{4} \times \frac{x^2-1}{4}\right)^{\frac{1}{2}} = \frac{2pqrsx}{(p^2-2q^2)(r^2-2s^2)} = \frac{2pqrs(p^2+2q^2)}{(p^2-2q^2)^2(r^2-2s^2)}.$$

Further, if twice the area be divided by either side, we shall have the perpendicular drawn to that side from its opposite angle. In order to have integral answers, both p^2-2q^2 and r^2-2s^2 must be $= \pm 1$.

The following sets of values of p, q, r, s will give such results:—

$\frac{p}{q}$, or $\frac{r}{s}$ =	1	1	3	7	17	41	99	239	577	1393
	0	1	2	5	12	29	70	169	408	985

and so on *ad infinitum*. The law of the series is as follows:—

"Multiply the terms of every set after the first by 2, and add the terms of the preceding set respectively for the succeeding set." We may sometimes, however, get integral answers, by assuming numbers not contained in the above series. When the answers are fractional, we can obtain integers by multiplying the results by the least common multiple of the denominators, and dividing by the greatest common measure of the numerators. But, except in a few cases, the triangles will have no two sides differing by unity.

It may be here remarked, that many of the answers to this problem will solve some of the conditions in Quest. 4840, *Reprint*, Vol. XXV., p. 105, as also other conditions not there specified.

In the above solution, p, q, r, s may be assumed any numbers; but when p, q are so taken that p^2-2q^2 is negative, r, s must be so taken that r^2-2s^2 shall be positive; for then $x, y, y+1$ will *all* be negative, but may be taken positive without error. It is also manifest that for every value of x (which depends upon the values of p, q) an infinite number of values of y and $y+1$ can be found, by varying infinitely the values of r, s.

Let $p=2,\ q=1,\ r=2,\ s=1$, then $x=3,\ y=4,\ y+1=5$;

" " $r=3,\ s=2$, " $x=3,\ y=25,\ y+1=26$;

" " $r=7,\ s=5$, " $x=3,\ y=148,\ y+1=149$, &c.

Let $p=3,\ q=2,\ r=6,\ s=1$, " $x=17,\ y=9,\ y+1=10$, least nos.;

" " $r=2,\ s=1$, " $x=17,\ y=25,\ y+1=26$;

" " $r=3,\ s=2$, " $x=17,\ y=144,\ y+1=145$, &c.

Let $p=2,\ q=1,\ r=3,\ s=1$, " $x=3,\ y=\frac{13}{7},\ y+1=\frac{20}{7}$;

or $\times 7$, " 21, 13, 20 are the sides.

Let $p=3,\ q=1,\ r=2,\ s=1$, " $x=\frac{11}{7},\ y=\frac{13}{7},\ y+1=\frac{20}{7}$;

or $\times 7$, " 11, 13, 20 are the sides.

It appears from *Reprint*, Vol. XXIII., p. 84, that 13, 14, 15 are the sides of the least scalene triangle whose sides consecutively differ by 1; and, from this Solution, that 9, 10, 17 are the sides of the least scalene triangle two of whose sides differ by 1; and also, that 11, 13, 20 are the sides of the least scalene triangle no two of which differ by 1; the areas being in all cases rational.

4224. (By Dr. HART.)—Find three numbers such that, if the product of every two be added to the sum of the same two, the sums shall be squares; and also, if the product of every two be added to the third, the sums shall be squares.

Solution by the PROPOSER

Here we have to make

$$xy+x+y=\square,\quad xz+x+z=\square,\quad yz+y+z=\square,$$
$$xy+z=\square,\quad xz+y=\square,\quad yz+x=\square.$$

Let $x = n^2$, and $y = (n+1)^2$, then we have

$$xy + x + y = n^2(n+1)^2 + n^2 + (n+1)^2 = (n^2+n+1)^2 = \square,$$

$$xz + x + z = n^2 z + n^2 + z = \square = (2n^2+n+2)^2;$$

whence we find $z = 4(n^2+n+1)$;

hence we have

$$yz + y + z = (n+1)^2 \times 4(n^2+n+1) + (n+1)^2 + 4(n^2+n+1) = (2n^2+3n+3)^2$$

$$xy + z = n^2(n+1)^2 \times 4(n^2+n+1) = (n^2+n+2)^2,$$

$$xz + y = n^2 \times 4(n^2+n+1) + (n+1)^2 = (2n^2+n+1)^2,$$

$$yz + x = (n+1)^2 \times 4(n^2+n+1) + n^2 = (2n^2+3n+2)^2;$$

so that all the conditions are fulfilled, n being any number.

Let $n=1$, then we have $x = 1$, $y = 4$, $z = 12$.

Let $n=2$, then we have $x = 4$, $y = 9$, $z = 28$, &c.

3192. (By J. M. Greenwood.)—The axes of two unequal cylinders intersect at right angles; determine the surface and volume common to both.

Solution by Artemas Martin, M.A.

Put r = radius of smaller cylinder, and R = radius of larger. Take the origin of coordinates at the intersection of the axes, the axis of the larger cylinder being the axis of x, and the axis of the smaller the axis of y; then the equations to the cylinder are

$$x^2 + z^2 = r^2, \quad y^2 + z^2 = R^2 \quad \text{.....(1, 2).}$$

Let S = surface of solid common to both cylinders,

S_1 = surface of smaller that is within the larger,

and S_2 = „ larger „ „ smaller;

then we have $S = S_1 + S_2$ .....(3).

But $$S_1 = 8\iint dx\,dy\left(1 + \frac{dz^2}{dx^2} + \frac{dz^2}{dy^2}\right)^{\frac{1}{2}} \quad \text{.....(4).}$$

From (1), $z = (r^2 - x^2)^{\frac{1}{2}}$, $\frac{dz}{dx} = -\frac{x}{(r^2-x^2)^{\frac{1}{2}}}$ and $\frac{dz}{dy} = 0$;

therefore $$S_1 = 8r\iint \frac{dx\,dy}{(r^2-x^2)^{\frac{1}{2}}} \quad \text{.....(5).}$$

Subtracting (1) from (2), we get $y = [R^2 - (r^2x^2)]^{\frac{1}{2}}$.

Integrating (5) with reference to y, we obtain

$$S_1 = 8r\int\frac{y\,dx}{(r^2-x^2)^{\frac{1}{2}}} = 8r\int_0^r \frac{dx\,[R^2-(r^2-x^2)]^{\frac{1}{2}}}{(r^2-x^2)^{\frac{1}{2}}}.$$

Let $$\frac{r}{R} = c, \quad \text{and} \quad rw = (r^2-x^2)^{\frac{1}{2}};$$

then we have $$x = r(1-w^2)^{\frac{1}{2}}, \quad dx = \frac{-rw\,dw}{(1-w^2)^{\frac{1}{2}}},$$

therefore $$S_1 = 8Rr\int_0^1 \frac{dw\,(1-c^2w^2)^{\frac{1}{2}}}{(1-w^2)^{\frac{1}{2}}} = 8Rr\,[E(c)],$$

and $$S_2 = 8\iint dx\,dz\left(1+\frac{dy^2}{dx^2}+\frac{dy^2}{dz^2}\right)^{\frac{1}{2}} \qquad \text{.....(6).}$$

But, from (2), $$y = (R^2-z^2)^{\frac{1}{2}}, \quad \frac{dy}{dz} = -\frac{z}{(R^2-z^2)^{\frac{1}{2}}}, \quad \frac{dy}{dx} = 0;$$

therefore $$S_2 = 8R\iint\frac{dx\,dz}{(R^2-z^2)^{\frac{1}{2}}} = 8r\int\frac{x\,dz}{(R^2-z^2)^{\frac{1}{2}}} = 8R\int_0^r\frac{dz\,(r^2-z^2)^{\frac{1}{2}}}{(R^2-z^2)^{\frac{1}{2}}}.$$

Let $z = rv$, then

$$S_2 = 8r^2\int_0^1\frac{(1-v^2)^{\frac{1}{2}}dv}{(1-c^2v^2)^{\frac{1}{2}}} = \frac{8r^2}{c^2}\int_0^1\frac{(1-c^2v^2)^{\frac{1}{2}}}{(1-v^2)^{\frac{1}{2}}} - \frac{8r^2}{c^2}\int_0^1\frac{(1-c^2)\,dv}{(1-v^2)^{\frac{1}{2}}(1-c^2w^2)^{\frac{1}{2}}}$$

$$= 8R^2[E(c)-(1-c^2)\,F(c)],$$

$$S = S_1+S_2 = (8R^2+8Rr)\,[E(c)]-8R^2(1-c^2)\,[F(c)].$$

Let V = volume of the solid common to both cylinders; then we have

$$V = 8\iiint dx\,dy\,dz = 8\iint y\,dx\,dz \qquad \text{.....(7).}$$

The limits of y are 0 and (R^2-z^2); of x, 0 and $(r^2-z^2)^{\frac{1}{2}}$; and of z, 0 and r; hence we have

$$V = 8\iint dx\,dz\,(R^2-z^2)^{\frac{1}{2}} = 8\int_0^r dz\,(r^2-z^2)^{\frac{1}{2}}(R^2-z^2)^{\frac{1}{2}}$$

$$= 8r^3R\int_0^1(1-v^2)^{\frac{1}{2}}(1-c^2v^2)^{\frac{1}{2}}dv = 8r^2R\int_0^1\frac{[1-(1+c^2)v^2+c^2v^4]\,dv}{(1-v^2)^{\frac{1}{2}}(1-c^2w^2)^{\frac{1}{2}}}$$

$$= \tfrac{8}{3}r^2R\left[v(1-v^2)^{\frac{1}{2}}(1-c^2v^2)^{\frac{1}{2}}\right]_0^1 + \tfrac{8}{3}r^2R\int_0^1\frac{[2-(1+c^2)v^2]\,dv}{(1-v^2)^{\frac{1}{2}}(1-c^2v^2)^{\frac{1}{2}}}$$

$$= \tfrac{8}{3}R^3(1+c^2)\int_0^1\frac{(1-c^2v^2)^{\frac{1}{2}}dv}{(1-v^2)^{\frac{1}{2}}} - \tfrac{8}{3}R^3(1-c^2)\int_0^1\frac{dv}{(1-v^2)^{\frac{1}{2}}(1-c^2v^2)^{\frac{1}{2}}}$$

$$= \tfrac{8}{3}R^3[(1+c^2)\,E(c)-(1-c^2)\,F(c)].$$

4915. (By Dr. COLLINS.)—Prove that (1) the equation of the parabolic asymptote of

$$ay^{n-2}x^2 + a'y^{n-3}x^3 + a''y^{n-4}x^4 + \&c. + by^{n-1} + b'y^{n-2}x + b''y^{n-3}x^2 + \&c. + cy^{n-2} + c'y^{n-3}x + \&c. = 0$$

is $$ax^2 + by + \frac{ab' - a'b}{a}x + c + \frac{b^2(aa'' - a'^2) + ab(a'b - ab'')}{a^3} = 0;$$

(2) the parabolic asymptote of third order of

$$ay^{n-3}x^3 + a'y^{n-4}x^4 + a''y^{n-5}x^5 + \&c. + by^{n-1} + b'y^{n-2}x + b'y^{n-3}x^2 + \&c. + cy^{n-2} + c'y^{n-3}x + c''y^{n-4}x^2 + \&c. + dy^{n-3} + d'y^{n-4}x + \&c. = 0$$

is $$a^3x^3 + a^2by^2 + a(ab - a'b)xy + \left\{a(ab'' - a''b) - a'(ab' - a'b)\right\}x^2 + Ay + Bx + C = 0;$$

(3) in the former case, when c is not $= 0$, the asymptotic common parabola above indicated has really a five-pointic contact at infinity with the given curve; and (4) find A, B, C in the second case (which is not done by Euler in pp. 98 and 102 of his *Analysis Infinitorum*) so that the cubic asymptote partly indicated above may really have a nine-point contact at infinity with the given curve of the nth order when b is not $= 0$; for when $b = 0$ in the first equation, the line projected to infinity is not a tangent to the curve, but a straight line passing through a double point of it; and when $b = 0$ in the second equation, then a tangent at a double point (and not at a point of inflection) has been projected to infinity.

Solution by Professor NASH, M.A.

As in my Solution on p. 38 of Vol. XXVI. of the *Reprint*, the equation of the curve may be written

$$ax^3 + (by^2 + b'yx + b''x^2 + \ldots + cy + c'x + \ldots + d)\left(1 + \frac{a'}{a}\frac{x}{y} + \frac{a''}{a}\frac{x^2}{y^2} + \frac{a'''}{a}\frac{x^3}{y^3} + \ldots\right)^{-1} = 0,$$

or $$ax^3 + by^2 + \frac{ab' - a'b}{a}yx + \frac{a(ab'' - a''b) - a'(ab' - a'b)}{a^2}x^2 + cy$$

$$+ \frac{ac' - a'c}{a}x + d + \frac{x^3}{y}\left\{b''' - \frac{a'b'}{a} - \frac{a''b'}{a} - \frac{a'''b}{a} + \frac{a'^2b'}{a^2} + \frac{2a'a''b}{a^2} - \frac{a'^3b}{a^3}\right\}$$

$$+ \frac{x^6}{y^2}\left\{b'''' - \frac{a'b'''}{a} - \frac{a''b''}{a} - \frac{a'''b'}{a} - \frac{a''''b}{a} + \frac{a'^2b''}{a^2} + \frac{a''^2b}{a^2} + \frac{2a'a''b'}{a^2}\right.$$

$$\left. + \frac{2a'a'''b}{a^2} - \frac{a'^3b'}{a^3} - \frac{3a'^2a''b}{a^3} + \frac{a'^4}{a^4}\right\} + \text{terms involving } \frac{x^5}{y^3}, \frac{x^2}{y}, \frac{y}{x} \text{ \&c.}$$

But if x, when infinite, be of the first order, y will be of the order $\frac{3}{2}$, $\frac{x^3}{y}$ will be of the order $\frac{1}{2}$, $\frac{x^4}{y^2}$ of the order 1, and $\frac{x^5}{y^3}$, $\frac{x^2}{y}$, $\frac{y}{x}$ will each be of the order $\frac{1}{2}$.

Now, in the equation of the asymptote there is no term of the order $\frac{1}{2}$, and therefore there must be no term of the order 0, therefore $c = 0$.

Writing α and β for the coefficients of $\frac{x^3}{y}$ and $\frac{x^4}{y^2}$ respectively, we find

the next approximation by substituting in

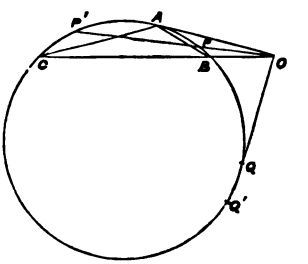
$\frac{x^3}{y}$ from $ax^3+by^2+\frac{ab'-a'b}{a}xy=0$, and in $\frac{x^4}{y^2}$ from $ax^3+by^2=0$.

Thus we get, for the equation of the asymptote,

$$ax^3+by^2+\frac{ab'-a'b}{a}yx+\frac{a(ab''-a''b)-a'(ab'-a'b)}{a^2}x^2$$
$$+y\left(c-\frac{b\alpha}{a}\right)+\frac{x}{a}\left\{ac'-a'c-\frac{\alpha}{a}(ab'-a'b)-b\beta\right\}=0.$$

3905. (By Professor WOLSTENHOLME, M.A.)—If A, B, C be three points on a circle, there are generally four points P, P′, Q, Q′ on the circle such that $PA^2 = PB \cdot PC$, &c., the straight lines PP′, QQ′ being parallel to the external and internal bisectors of the angle A, and meeting the straight line BC in the same point as the tangent at A. The points Q, Q′ will coincide when $a^2 = 4bc$; and in this case the locus of Q (B, C being fixed points) is a rectangular hyperbola of which B, C are the foci, and the loci of P, P′ are two circular cubics whose foci are B, C and one vertex of the rectangular hyperbola. These cubics are inverses of the rectangular hyperbola with respect to its vertices. The points Q, Q′ are impossible if $a^2 < 4bc$; but if $a^2 = 4n \cdot bc$, where $n > 1$, the loci of P, P′, Q, Q′ are all circular cubics having two foci at B, C.

Solution by the PROPOSER.



At a point P on the circle (between A and B) the areal coordinates (to the triangle ABC) are as $PB \cdot PC \sin A$, $-PC \cdot PA \sin B$, $PA \cdot PB \sin C$; hence, when $PA^2 = PB \cdot PC$, we shall have $\frac{x^2}{a^2} = \frac{-yz}{bc}$; and the same equation will hold at P′ (a point between A and C).

But $a^2yz + b^2zx + c^2xy = 0$,

or $bcx^2 = b^2zx + c^2xy$,

so that the equation of PP′ is

$$bcx = b^2z + c^2y,$$

which is parallel to $bz + cy = 0$, and meets BC in O, the point where $b^2z + c^2y = 0$, or where the tangent at A meets it.

So for the points Q, Q′ (on the opposite side of BC to A), $\frac{x^2}{a^2} = \frac{yz}{b^2}$, and the equation of QQ′ is $bcx + b^2z + c^2y = 0$, which is parallel to $bz - cy = 0$, and passes through O.

Now suppose $a^2 = 4nbc$; then at P, P′ we have

$$bcx = b^2z + c^2y, \quad 4nbcyz + x(b^2z + c^2y) = 0,$$

whence $$4nb^2c^2yz + (b^2z + c^2y) = 0,$$

or $$\frac{y}{z} = -bc \cdot \frac{(2n+1)\,bc \pm \{(2n+1)^2 b^2c^2 - b^2c^2\}^{\frac{1}{2}}}{c^4}$$

$$= -b^2 \cdot \frac{2n+1 \pm 2\{n(n+1)\}^{\frac{1}{2}}}{c^2};$$

or, at P, $\frac{y}{z} = \frac{-b^2\{(n+1)^{\frac{1}{2}} + n^{\frac{1}{2}}\}}{c^2\{(n+1)^{\frac{1}{2}} - n^{\frac{1}{2}}\}}$; at P', $\frac{y}{z} = \frac{-b^2\{(n+1)^{\frac{1}{2}} - n^{\frac{1}{2}}\}}{c^2\{(n+1)^{\frac{1}{2}} + n^{\frac{1}{2}}\}}$;

or the coordinates of P are

$$\frac{2n^{\frac{1}{2}}bc}{\lambda}, \quad \frac{b^2\{(n+1)^{\frac{1}{2}} + n^{\frac{1}{2}}\}}{\lambda}, \quad \frac{c^2\{n^{\frac{1}{2}} - (n+1)^{\frac{1}{2}}\}}{\lambda},$$

where $$\lambda = (b+c)^2 n^{\frac{1}{2}} + (n+1)^{\frac{1}{2}}(b^2 - c^2).$$

Let (r_1, r_2) denote BP, CP; then

$$-r_1^2 = a^2(y-1)z + b^2zx + c^2(y-1)x = -(a^2z + c^2x),$$

since $$a^2yz + b^2zx + c^2xy = 0;$$

or $$r_1^2 = \frac{a^2c^2\{n^{\frac{1}{2}} - (n+1)^{\frac{1}{2}}\} + \frac{c^2a^2}{2n^{\frac{1}{2}}}}{\lambda} = \frac{a^2c^2}{2\lambda n^{\frac{1}{2}}}\{(n+1)^{\frac{1}{2}} - n^{\frac{1}{2}}\}^2.$$

So $$r_2^2 = a^2y + b^2x = \frac{a^2b^2}{2\lambda n^{\frac{1}{2}}}\{(n+1)^{\frac{1}{2}} + n^{\frac{1}{2}}\}^2.$$

Hence $$AB \cdot PB : AC \cdot PC = (n+1)^{\frac{1}{2}} - n^{\frac{1}{2}} : (n+1)^{\frac{1}{2}} + n^{\frac{1}{2}}.$$

Let $(n+1)^{\frac{1}{2}} + n^{\frac{1}{2}} = k$; then $\lambda = kb^2 - \frac{c^2}{k} + \frac{a^2}{2n^{\frac{1}{2}}}$,

and we have the two equations

$$r_1^2\left[2n^{\frac{1}{2}}\left(kb^2 - \frac{c^2}{k}\right) + a^2\right] = \frac{a^2c^2}{k^2},$$

or $$kb^2 \cdot 2n^{\frac{1}{2}}r_1^2 - \frac{c^2}{k}\left(2n^{\frac{1}{2}}r_1^2 + \frac{a^2}{k}\right) = -a^2r_1^2;$$

$$r_2^2\left[2n^{\frac{1}{2}}\left(kb^2 - \frac{c^2}{k}\right) + a^2\right] = a^2b^2k^2,$$

or $$kb^2(2n^{\frac{1}{2}}r_2^2 - ka^2) - \frac{c^2}{k} \cdot 2n^{\frac{1}{2}}r_2^2 = -a^2r_2^2;$$

so that $$kb^2\left\{4nr_1^2r_2^2 - (2n^{\frac{1}{2}}r_2^2 - ka^2)\left(2n^{\frac{1}{2}}r_1^2 + \frac{a^2}{k}\right)\right\}$$

$$= a^2\left\{r_2^2\left(2n^{\frac{1}{2}}r_1^2 + \frac{a^2}{k}\right) - r_1^2r_2^2 \cdot 2n^{\frac{1}{2}}\right\},$$

or $$kb^2\left(2n^{\frac{1}{2}}r_1^2k - 2n^{\frac{1}{2}}\frac{r_2^2}{k} + a^2\right) = \frac{a^2r_2^2}{k},$$

and similarly $$\frac{c^2}{k}\left(2n^{\frac{1}{2}}r_1^2k - 2n^{\frac{1}{2}}\frac{r_2^2}{k} + a^2\right) = ka^2r_1^2;$$

whence $$\left(2n^{\frac{1}{2}}r_1^2k - 2n^{\frac{1}{2}}\frac{r^2}{k} + a^2\right)^2 = 16n^2r_1^2r_2^2,$$

or $$2n^{\frac{1}{2}}\left(r_1^2k - \frac{r_2^2}{k}\right) + a^2 = \pm 4nr_1r_2, \text{ giving the locus of P.}$$

Now suppose another focus of this curve to be the point $(0, m, 1-m)$, and let r_3 be the distance of P from this point; then we must have an equation of the form $\lambda r_1 + \mu r_2 = r_3$, and we have identically

$$mr_1^2 + (1-m)r_2^2 - r_3^2 - a^2m(1-m) = 0,$$

or $$mr_1^2 + (1-m)r_2^2 - (\lambda r_1 + \mu r_2)^2 - a^2m(1-m) = 0,$$

which must coincide with our former equation in r_1, r_2. Hence we have, to determine λ, μ, m, the three equations

$$\frac{\lambda^2 - m}{2n^{\frac{1}{2}}\{(n+1)^{\frac{1}{2}} + n^{\frac{1}{2}}\}} = \frac{\mu^2 - (1-m)}{2n^{\frac{1}{2}}\{n^{\frac{1}{2}} - (n+1)^{\frac{1}{2}}\}} = \frac{\lambda\mu}{\pm 2n} = m(1-m);$$

whence

$$\lambda^2\mu^2 = 4n^2m^2(1-m)^2 = \left\{m + 2n^{\frac{1}{2}}km(1-m)\right\}\left\{1-m-\frac{2n^{\frac{1}{2}}}{k}m(1-m)\right\},$$

or (two roots being of course 0 and 1)

$$4n^2m(1-m) = \left\{1 + 2n^{\frac{1}{2}}k(1-m)\right\}\left\{1 - \frac{2n^{\frac{1}{2}}}{k}m\right\},$$

or $$4n(n+1)m(1-m) = 1 + 2n^{\frac{1}{2}}k(1-m) - \frac{2n^{\frac{1}{2}}}{k}m$$

$$= 1 + 2n^{\frac{1}{2}}\{(n+1)^{\frac{1}{2}} + n^{\frac{1}{2}}\} - 2n^{\frac{1}{2}}.m.2(n+1)^{\frac{1}{2}}$$

$$= \{(n+1)^{\frac{1}{2}} + n^{\frac{1}{2}}\}^2 - 4m\{n(n+1)\}^{\frac{1}{2}},$$

or $$m^2.4n(n+1) - 4m\{n(n+1) + [n(n+1)]^{\frac{1}{2}}\} + \{(n+1)^{\frac{1}{2}} + n^{\frac{1}{2}}\}^2 = 0;$$

$$\therefore\ 2m[n(n+1)]^{\frac{1}{2}} = [n(n+1)]^{\frac{1}{2}} \pm \left[\{[n(n+1)]^{\frac{1}{2}} + 1\}^2 - [(n+1)^{\frac{1}{2}} + n^{\frac{1}{2}}]^2\right]^{\frac{1}{2}}$$

$$= 1 + [n(n+1)]^{\frac{1}{2}} \pm [n(n-1)]^{\frac{1}{2}};$$

and therefore $$2(1-m)[n(n+1)]^{\frac{1}{2}} = [n(n+1)]^{\frac{1}{2}} - 1 \mp [n(n-1)]^{\frac{1}{2}},$$

so that $$\lambda\mu = \frac{1}{2(n+1)}\left(n(n+1) - \{1 \pm [n(n-1)]^{\frac{1}{2}}\}^2\right) = \frac{[n^{\frac{1}{2}} \mp (n-1)^{\frac{1}{2}}]^2}{2(n+1)}.$$

Taking the upper sign, we get

$$\lambda^2 = m\left\{1 + 2n^{\frac{1}{2}}[(n+1)^{\frac{1}{2}} + n^{\frac{1}{2}}](1-m)\right\}$$

$$= \frac{1 + [n(n+1)]^{\frac{1}{2}} + [n(n-1)]^{\frac{1}{2}}}{2[n(n+1)]^{\frac{1}{2}}} + \frac{2n^{\frac{1}{2}}[(n+1)^{\frac{1}{2}} + n^{\frac{1}{2}}][n^{\frac{1}{2}} - (n-1)^{\frac{1}{2}}]^2}{4n(n+1)}$$

$$= \frac{1}{2(n+1)}\left\{3n + 2[n(n+1)]^{\frac{1}{2}} - (n^2-1)^{\frac{1}{2}} - 2[n(n-1)]^{\frac{1}{2}}\right\},$$

or $$\lambda = \pm\left\{\frac{2n^{\frac{1}{2}} + (n+1)^{\frac{1}{2}} - (n-1)^{\frac{1}{2}}}{2(n+1)^{\frac{1}{2}}}\right\}.$$

Similarly, $$\mu = \pm\left\{\frac{-2n^{\frac{1}{2}} + (n+1)^{\frac{1}{2}} + (n-1)^{\frac{1}{2}}}{2(n+1)^{\frac{1}{2}}}\right\};$$

so that $\pm\lambda \pm \mu = 1$, as ought to be the case.

The locus of P is thus completely determined; and that of P′ only differs from it by interchanging B and C. We observe that the two axial foci of these loci (other than B and C) are impossible except when $n>1$; and that they coincide when $n=1$.

Next, at Q, Q′ we have $bcx+b^2z+c^2y=0$, or $4nb^2c^2yz-(b^2z+c^2y)^2=0$, an equation differing from that for P, P′ only by putting $-n$ for n; we shall thus get

$$\frac{y}{z}=-\frac{b^2}{c^2}\{1-2n\pm[2n(n-1)]^{\frac{1}{2}}\}=\frac{b^2}{c^2}\cdot\frac{n^{\frac{1}{2}}\pm(n-1)^{\frac{1}{2}}}{n^{\frac{1}{2}}\mp(n-1)^{\frac{1}{2}}};$$

or at Q the coordinates are

$$\frac{-2n^{\frac{1}{2}}bc}{\lambda},\quad \frac{b^2[n^{\frac{1}{2}}+(n-1)^{\frac{1}{2}}]}{\lambda},\quad \frac{c^2[n^{\frac{1}{2}}-(n-1)^{\frac{1}{2}}]}{\lambda};$$

and $$r_1^2=a^2z+c^2x=\frac{a^2c^2}{\lambda}\left(n^{\frac{1}{2}}-(n-1)^{\frac{1}{2}}-\frac{1}{2n^{\frac{1}{2}}}\right)=\frac{a^2c^2}{2n^{\frac{1}{2}}\lambda}[n^{\frac{1}{2}}-(n-1)^{\frac{1}{2}}]^2,$$

so $$r_2^2=\frac{a^2b^2}{2n^{\frac{1}{2}}\lambda}[n^{\frac{1}{2}}+(n-1)^{\frac{1}{2}}]^2.$$

Now, when $n=1$, $\lambda=(b-c)^2$, $r_1=\frac{ac}{(2\lambda)_{\frac{1}{2}}}$, $r_2=\frac{ab}{(2\lambda)^{\frac{1}{2}}}$;

therefore $r_2\sim r_1=\frac{a}{2^{\frac{1}{2}}}$; or the locus is a rectangular hyperbola whose foci are B, C.

Generally, $\lambda=b^2k+\frac{c^2}{k}-\frac{a^2}{2n^{\frac{1}{2}}}$, $\therefore\ r_1^2\left\{2n^{\frac{1}{2}}\left(b^2k+\frac{c^2}{k}\right)-a^2\right\}=\frac{a^2c^2}{k^2}$,

or $$b^2k(2n^{\frac{1}{2}}r_1^2)+\frac{c^2}{k}\left(2n^{\frac{1}{2}}r_1^2-\frac{a^2}{k}\right)=a^2r_1^2,$$

and $$b^2k(2n^{\frac{1}{2}}r_2^2-a^2k)+\frac{c^2}{k}(2n^{\frac{1}{2}}r_2^2)=a^2r_2^2,$$

or $$b^2k\left\{4nr_1^2r_2^2-\left(2n^{\frac{1}{2}}r_1^2-\frac{a^2}{k}\right)(2n^{\frac{1}{2}}r_2^2-a^2k)\right\}$$
$$=a^2\left\{2n^{\frac{1}{2}}r_1^2r_2^2-r_2^2\left(2n^{\frac{1}{2}}r_1^2-\frac{a^2}{k}\right)\right\},$$

or $$b^2k\left\{2n^{\frac{1}{2}}\left(kr_1^2+\frac{r_2^2}{k}\right)-a^2\right\}=\frac{a^2r_2^2}{k},$$

and $$\frac{c^2}{k}\left\{2n^{\frac{1}{2}}\left(kr_1^2+\frac{r_2^2}{k}\right)-a^2\right\}=a^2r_1^2k,$$

whence $$2n^{\frac{1}{2}}\left(kr_1^2+\frac{r_2^2}{k}\right)-a^2=\pm4nr_1r_2,$$

where $k=n^{\frac{1}{2}}+(n-1)^{\frac{1}{2}}$ is the locus of Q.

With a notation as in the former case, we get

$$\frac{\lambda^2-m}{2n^{\frac{1}{2}}[n^{\frac{1}{2}}+(n-1)^{\frac{1}{2}}]}=\frac{\mu^2-(1-m)}{2n^{\frac{1}{2}}[n^{\frac{1}{2}}-(n-1)^{\frac{1}{2}}]}=\frac{\lambda\mu}{\pm2n}=-m(1-m),$$

whence

$$4n^2m(1-m)=\{1-(1-m)2n^{\frac{1}{2}}[n^{\frac{1}{2}}+(n-1)^{\frac{1}{2}}]\}\{(1-m)2n^{\frac{1}{2}}[n^{\frac{1}{2}}-(n-1)^{\frac{1}{2}}]\},$$
$$4n(n-1)m(1-m)=1-2n^{\frac{1}{2}}[n^{\frac{1}{2}}+(n-1)^{\frac{1}{2}}]+4m[n(n-1)]^{\frac{1}{2}},$$
$$4m^2n(n-1)-4m\{n(n-1)-[n(n-1)]^{\frac{1}{2}}\}-[n^{\frac{1}{2}}+(n-1)^{\frac{1}{2}}]^2=0,$$

$$\text{or } 2m[n(n-1)]^{\frac{1}{2}} = -1+[n(n-1)]^{\frac{1}{2}}+\left[\left\{1-[n(n-1)]^{\frac{1}{2}}\right\}^2+[n+(n-1)^{\frac{1}{2}}]^2\right]^{\frac{1}{2}}$$
$$= -1+[n(n-1)]^{\frac{1}{2}} \pm [n(n+1)]^{\frac{1}{2}}.$$

Taking the upper sign, we have

$$\lambda^2 = m\left\{1-2n^{\frac{1}{2}}[n^{\frac{1}{2}}+(n-1)^{\frac{1}{2}}](1-m)\right\}$$
$$= \frac{-1+[n(n-1)]^{\frac{1}{2}}+[n(n+1)]^{\frac{1}{2}}}{2[n(n-1)]^{\frac{1}{2}}} + \frac{2n^{\frac{1}{2}}[n^{\frac{1}{2}}+(n-1)^{\frac{1}{2}}]}{4n(n-1)}[(n+1)^{\frac{1}{2}}-n^{\frac{1}{2}}]^2$$
$$= \frac{1}{2(n-1)}\left\{3n+2[n(n-1)]-(n^2-1)-2[n(n+1)]^{\frac{1}{2}}\right\},$$

or
$$\lambda = \pm\left\{\frac{2n^{\frac{1}{2}}+(n-1)^{\frac{1}{2}}-(n+1)^{\frac{1}{2}}}{2(n-1)^{\frac{1}{2}}}\right\}.$$

Similarly we have $\mu = \pm\left\{\frac{-2n^{\frac{1}{2}}+(n-1)^{\frac{1}{2}}+(n+1)^{\frac{1}{2}}}{2(n-1)^{\frac{1}{2}}}\right\}$,

so that $\pm\lambda \pm \mu = 1$.

When $n=1$, the two extra foci are at infinity, and the locus of Q, Q′ has been seen to be the rectangular hyperbola of which B, C are foci. In the general case, the loci are two different circular cubics having the common foci B, C; and if O, Q, Q′ be the order, a third focus is a point K dividing CB produced in the ratio

$$-1+[n(n-1)]^{\frac{1}{2}}+[n(n+1)]^{\frac{1}{2}} : 1+[n(n-1)]^{\frac{1}{2}}-[n(n+1)]^{\frac{1}{2}},$$

or BC produced in the ratio

$$[n(n+1)]^{\frac{1}{2}}+[n(n-1)]^{\frac{1}{2}}+1 : [n(n+1)]^{\frac{1}{2}}-[n(n-1)]^{\frac{1}{2}}+1;$$

and if $(r_1 r_2 r_3)$ be the distances of any point Q from B, C, K,

$$[2n^{\frac{1}{2}}+(n-1)^{\frac{1}{2}}-(n+1)^{\frac{1}{2}}]r_1+[-2n^{\frac{1}{2}}+(n-1)^{\frac{1}{2}}+(n+1)^{\frac{1}{2}}]r_2 = 2(n-1)^{\frac{1}{2}}r_3.$$

The locus of Q′ is found from that of Q by interchanging B and C.

In the case when $n=1$, the double focus of the locus of P divides CB in the ratio $2^{\frac{1}{2}}+1 : 2^{\frac{1}{2}}-1$, so that it is the vertex of the rectangular hyperbola. It is easy to prove that in this case the loci of P, P′ are inverses of the rectangular hyperbola with respect to its vertices, but I think it will be found that P, Q are *not* inverse points. I have mislaid the note of my results; but, according to my recollection, if L, L′ be the vertices (nearer and farther from B), P′Q subtends a right angle at L′, and PQ a right angle at L.

5255. (By C. W. Merrifield, F.R.S.)—Show that

$$a_n(a_0, a_1, \ldots a_{2n}\,\mathcal{J}\,x, y)^{2n}$$
$$\equiv \left\{a_0x^n-a_n(-y)^n\right\}\left\{a_nx^n-a_{2n}(-y)^n\right\}$$
$$+2nxy\left\{a_1x^{n-1}-a_n(-y)^{n-1}\right\}\left\{a_nx^{n-1}-a_{2n-1}(-y)^{n-1}\right\}+\ldots+Qx^n(-y)^n,$$

where Q is the quadrinvariant.

Solution by the PROPOSER.

The quadrinvariant may be written in the form

$$(a_0 a_{2n} - a_n^2) - 2n\,(a_1 a_{2n-1} - a_n^2) + \ldots\ldots,$$

the last term being $(-)^{n-1}\,\dfrac{2n \,.\, 2n-1 \,\ldots\ldots\, n+2}{1 \,.\, 2 \,.\, 3 \,\ldots\ldots\, n-1}\,(a_{n-1} a_{n+1} - a_n^2)$.

Remark that $B(Ax^2 + 2Bxy + Cy^2) \equiv (Ax + By)(Bx + Cy) - (AC - B^2)\,xy$, and that $B(Ax^2 - 2Bxy + Cy^2) \equiv (Ax - By)(Bx - Cy) + (AC - B^2)\,xy$; then the proposition follows at once.

5400. (By J. W. L. GLAISHER, F.R.S.)—Verify that

$$u = \int_{a^{\frac{1}{2}}x^{\frac{1}{2}}}^{\infty} e^{-x^n - \frac{a^n x^n}{x^n}}\,dx$$

is a particular integral of the differential equation

$$\frac{d^2u}{dx^2} - n^2 a^n x^{n-2}\,u = \tfrac{1}{2} e^{-2a^{\frac{1}{2}n} x^{\frac{1}{2}n}} \,.\, a^{\frac{1}{2}} x^{-\frac{3}{2}},$$

which is a particular case of the extended Riccati's equation

$$\frac{d^2u}{dx^2} - x^b u = f(x).$$

Solution by ROBERT RAWSON.

This is a particular case of the more general theorem proposed as Question 5440, which is investigated in my note-book, dated 1874. Taking, in that more general theorem, $x_1 = -a^n x^n$, $c = 0$, $c_2 = 1$, $nc_3 = -1$, $c_1 n = -n - 1$, $\phi(z) = z^{-n}$, $\alpha = \infty$, $\beta = a^{\frac{1}{2}} x^{\frac{1}{2}}$, and observing that $e^{-a^n - \frac{a^n x^n}{x^n}}$ is not infinite when $\alpha = \infty$, the theorem in the question readily follows.

5394. (By Professor MINCHIN, M.A.)—If a curve and its inverse be described with the same law of force, viz., $\mu r^5 p^{-5}$; prove that both curves are included in the class whose equation is $r^6 + ap^3 + b = 0$.

Solution by R. F. DAVIS, B.A.; Rev. H. M. DAY, M.A.; *and others.*

If it be possible for a curve and its inverse to be described with the same law of central force, $\lambda p^{-3}\dfrac{dp}{dr}$ must $= \mathrm{P}^{-3}\dfrac{d\mathrm{P}}{d\mathrm{R}}$, the capital letters re-

ferring to the inverse, and λ being a constant. The connecting relations are of course $rR = \kappa^2$, and $p : r = P : R$, or $P = p\kappa^2 r^{-2}$. Now $\frac{dR}{dr} = -\frac{R}{r} = -\frac{P}{p}$; hence $\lambda p^{-4}\frac{dp}{dr} + P^{-4}\frac{dP}{dr} = 0$. Integrating, $\lambda p^{-3} + P^{-3} = \text{const.}$, and substituting for P from above, a relation is obtained of the form $r^6 + ap^3 + b = 0$; also $p^{-3}\frac{dp}{dr}$, which defines the force, $\propto r^5 p^{-5}$.

5317. (By S. Roberts, M.A.)—Prove that the locus of the poles of the tangents of a cubic curve $U = 0$ is

$$TU^3 - 20U^2H + 28H^3 = 0,$$

where H is the Hessian and S, T are the usual invariants.

Solution by the Proposer.

Since the polar line is to touch the curve, we obtain the general locus by substituting $\frac{du}{dx}, \frac{du}{dy}, \frac{du}{dz}$ for x, y, z in the reciprocal. The curve itself is a factor of the locus. For the canonical form $ax^3 + by^3 + cz^3 + 6dxyz = 0$, the coefficient of x^{12} in the reciprocal after the substitution is $a^6b^2c^2$.

Dividing by a on account of the factor U, we have to account for $a^5b^2c^2$.

Now $S = d^4 - dabc$, $T = a^2b^2c^2 - 20abcd^3 - 8d^6$, $H = d^2ax^3 + \&c.$

Then
$$TU^3 = (a^5b^2c^2 - 20a^4bcd^3 - 8a^3d^6)\,x^9 + \ldots,$$
$$-20\,SU^2H = (20a^4bcd^3 - 20a^3d^6)\,x^9 + \ldots,$$
$$28H^3 = 28a^3d^6\,x^9 + \ldots,$$

and the result is $TU^3 - 20SU^2H + 28H^3 = 0$.

Hence, $T = 0$ being the condition that the second Hessian may be the original curve, it appears that then the first Hessian is the locus of a pole of the tangents of the curve, since H becomes a factor of the locus.

5190. (By J. J. Walker, M.A.)—Referring to Professor Townsend's Quest. 5133, if ρ is the ratio of the density of the cone to that of the fluid in a case in which it can float in free equilibrium, with its axis inclined to the vertical at an angle ϕ, and if θ is the semi-angle of the cone, prove (1) that $\cos^2\phi = \frac{\sin^2\theta}{1 - \rho^{\frac{4}{3}}\sec^2\theta}$, and (2) that the base will be only just clear of the water when $\rho = (\cos 2\theta)^{\frac{3}{4}}$.

Solution by the PROPOSER.

1. Referring to the figure in my solution of Quest. 5133 (*Reprint*, Vol. XXVII., p. 57), let AN be the perpendicular on DE. Then we have

$$\cos^2\phi = \frac{AN^2}{AF^2} = \frac{AD\,.\,AE\sin^2\theta}{DF\,.\,FE} = \frac{AF\,.\,AQ\sin^2\theta}{AF\,.\,FQ},$$

or
$$\sin^2\theta = \frac{AF\,.\,FQ\cos^2\phi}{AF\,.\,AQ};$$

that is,
$$\sin^2\theta = \left(1-\frac{AF^2}{AF\,.\,FQ}\right)\cos^2\phi = \left(1-\frac{AF^2}{AD\,.\,AE}\right)\cos^2\phi$$
$$= \left(1-\frac{AD\,.\,AE}{AQ^2}\right)\cos^2\phi = \left(1-\frac{AK^2}{AQ^2}\right)\cos^2\phi$$
$$= \left(1-\frac{AM^2\sec^2\theta}{AQ^2}\right)\cos^2\phi = (1-\rho^{\frac{2}{3}}\sec^2\theta)\cos^2\phi.$$

2. When C and E coincide, $\phi = QCD = QAD = \theta$, and the above relation becomes $\cos^2\theta - \rho^{\frac{2}{3}} = \sin^2\theta$, whence $\rho = (\cos 2\theta)^{\frac{3}{2}}$.

In order that ϕ should have a sensible magnitude, $1-\rho^{\frac{2}{3}}\sec^2\theta$ must be greater than $\sin^2\theta$, that is, than $1-\cos^2\theta$; or $\rho^{\frac{2}{3}}\sec^2\theta$ must be less than $\cos^2\theta$, or ρ less than $\cos^6\theta$. Thus, if $\theta = 30°$, ρ must be less than $\frac{27}{64}$ in order that an inclined position of floatation should be possible, but not less than $\frac{1}{2\sqrt{2}}$ in order that the base may be entirely unsubmerged.

5281. (By W. S. B. WOOLHOUSE, F.R.A.S.) — Logarithmic tables always give the nearest figure, so that the error in the last digit is always comprised within the limits $\pm\frac{1}{2}$. When any number (n) of logarithms are taken promiscuously and added together, determine the probability that the error of the sum will be contained within the limits $\pm k$, k being any given value less than $\frac{1}{2}n$.

Solution by HUGH MCCOLL, B.A.

Let the errors of the respective n logarithms be x, y, z, &c.; and let q = the chance that $x+y+z+\ldots\ldots$ will be greater than k; then the *required chance* $= 1-2q$.

Let A = the statement $p'(x-\frac{1}{2})\,p(x+\frac{1}{2})\,p'(y-\frac{1}{2})\,p(y+\frac{1}{2})\ldots\ldots$, and let Q = the statement $p(x+y+z+\ldots\ldots-k)$; then we have

$$q = \frac{AQ\int dx\int dy\int dz\ \ldots\ldots}{A\int dx\int dy\int dz\ \ldots\ldots},$$

the statements to the left of the integrals fixing the limits of integration.

The denominator of the above fraction is easily seen to be

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} dx \int_{-\frac{1}{2}}^{\frac{1}{2}} dy \int_{-\frac{1}{2}}^{\frac{1}{2}} dz \ldots\ldots = 1,$$

so that

$$q = AQ \int dx \int dy \int dz \ldots\ldots$$

Taking n successively = 2, 3, 4, &c., and applying rules 3, 4, 5 of my *Symbolical Language*, for determining the limits of the variables x, y, z, &c., and of the constant k, we arrive finally at the following general result:

When k is between the limits $\frac{1}{2}n$ and $\frac{1}{2}n-1$; and, again, between $\frac{1}{2}n-1$ and $\frac{1}{2}n-2$; we have [putting $n!$ for $1.2.3 \ldots n$],

$$q = \frac{1}{n!}(\tfrac{1}{2}n-k)^n; \quad q = \frac{1}{n!}\left\{(\tfrac{1}{2}n-k)^n - n(\tfrac{1}{2}n-k-1)^n\right\};$$

and generally, when k is between the limits $\frac{1}{2}n-r$ and $\frac{1}{2}n-r-1$,

$$q = \frac{1}{n!}\left\{a^n - n(a-1)^n + \frac{n(n-1)}{1.2}(a-2)^n - \ldots + (-1)^r \frac{n!}{r!\,n-r!}(a-r)^n\right\}$$

in which $a = \frac{1}{2}n-k$, and r is any integer less than a.

As a particular case let $n = 5$, and let $k = \frac{1}{2}$; then $1-2q = .55$. The integrations may be abbreviated by suitably changing the variables.

5295. (By J. F. Moulton, M.A.)—An indefinitely thin shell in the form of an infinite elliptical cylinder, the section of the two surfaces being similar, similarly situated, and concentric curves, is surrounded by homogeneous fluid which it attracts. Show that the pressure at any point is given by $2m\rho \log \frac{a+b}{a'+b'}$, where a', b', a, b are the semi-axes of the confocal cylinders through the point and bounding the fluid respectively.

Solution by Professor Townsend, F.R.S.

The attraction of the cylinder at any point, external or internal, being, as is well known, the same, for the law of the inverse square of the distance, as if twice the mass of its unit of length were distributed, by uniform longitudinal condensation all through, over the annular area of its transverse section by the plane through the point perpendicular to its axis, and attracted, with the same absolute force, according to the law of the inverse simple distance instead; hence, denoting by m the mass of its unit of length, by f the absolute force of its attraction, by a and b the semi-axes of its transverse elliptic section, by λ the parameter of the confocal cylinder through any point of the attracted fluid, by V the potential of its attraction for the same point, by ρ the density of the fluid, and by p the fluid pressure at the point; then, since, for any point P of the fluid,

$$dp = \rho d\mathrm{V}, \quad \text{and} \quad d\mathrm{V} = -2mf\rho \frac{\lambda d\lambda}{(a^2+\lambda^2)^{\frac{1}{2}}(b^2+\lambda^2)^{\frac{1}{2}}},$$

therefore, for any two points P_1 and P_2 of its mass,

$$p_1 - p_2 = 2mf\rho \,.\, \log \frac{(a^2+\lambda_2^2)^{\frac{1}{2}} + (b^2+\lambda_2^2)^{\frac{1}{2}}}{(a^2+\lambda_1^2)^{\frac{1}{2}} + (b^2+\lambda_1^2)^{\frac{1}{2}}}, \text{ and therefore \&c.}$$

Cor.—When $a=b$, that is, when the cylinder is circular, and when consequently its attraction is the same at every external point as if its mass were concentrated uniformly along its axis, then, from the above,

$$p_1 - p_2 = 2mf\rho\,(\log r_2 - \log r_1),$$

where r_1 and r_2 are the distances of P_1 and P_2 from the axis; a result easily obtained at once independently for that simple case.

5356. (By A. Winslow.)—If any three circles touch each other, show that the three lines joining the centre of each circle to the point of tangency of the other two intersect in the same point.

Solution by Elizabeth Blackwood.

Let X, Y, Z be the three centres; XYZ the triangle of reference; x, y, z the perpendiculars from any point upon the sides YZ, ZX, XY respectively; and a, b, c the three radii. It may be easily shown that the equations to the three lines are respectively

$yb \sin Y - zc \sin Z = 0$, $zc \sin Z - xa \sin X = 0$, $xa \sin X - yb \sin Y = 0$;

and since any one of these equations follows as a necessary consequence of the other two, any point in two of the lines must also be in the third line; in other words, the three lines pass through the same point.

5288. (By the Rev. A. F. Torry, M.A.)—Taking the eccentricity of the earth's orbit round the sun to be $\frac{1}{60}$, and the year to consist of $365\frac{1}{4}$ days; determine the greatest and least angular velocities of the earth about the sun.

Solution by R. Leidhold, B.A.; Prof. Evans, M.A., *and others.*

Let e be the given eccentricity, and a the semi-axis major of the orbit; then, if the orbit be referred to its major axis (the sun's centre being the origin), we have for any vectorial angle θ the relation

$$\tan \tfrac{1}{2}\theta = \left(\frac{1+e}{1-e}\right)^{\frac{1}{2}} \tan \tfrac{1}{2}\phi,$$

where ϕ is the corresponding eccentric angle [see Salmon's *Conics*, 5th Ed., p. 209, Ex. 13]. Denote the sector bounded by the orbit, the radius vector

of θ, and the initial line, by $S = \frac{1}{2}a^2(1-e^2)^{\frac{1}{2}}(\phi - e\sin\phi)$ [see Schlömilch's *Compendium*, Band I., p. 378];

therefore $dS = \frac{1}{2}a^2(1-e^2)^{\frac{1}{2}}(1-e\cos\phi)\,d\phi$, $d\theta = \dfrac{(1-e^2)^{\frac{1}{2}}}{1-e\cos\phi}\,d\phi$.

Therefore $\dfrac{dS}{dt} = \frac{1}{2}a^2(1-e\cos\phi)^2\dfrac{d\theta}{dt} = \dfrac{\pi a^2(1-e^2)^{\frac{1}{2}}}{365\frac{1}{4}}$;

whence the angular velocity, or $\dfrac{d\theta}{dt} = \dfrac{2\pi}{365\frac{1}{4}}\cdot\dfrac{(1-e^2)^{\frac{1}{2}}}{(1-e\cos\phi)^2}$,

which is a maximum when $\phi = 0$, and a minimum when $\phi = \pi$.

5165. (By R. A. Roberts, M.A.)—Prove that the envelop of the director circles of a series of conics passing through four fixed points, is a bicircular quartic of which the vertices of the fixed triangle that is self-conjugate with regard to the conics are foci.

I. *Solution by* Professor Townsend, F.R.S.

The envelope of a variable circle whose centre moves on a fixed conic, and which intersects at right angles a fixed circle, being a bicircular quartic of which the intersections of the conic and circle are a tetrad of concyclic foci (see Casey's *Bicircular Quartics*, fundamental definition); the above pretty result follows, in consequence, immediately from the two well known properties, that (1) the centre of a variable conic passing through four fixed points, and therefore that of its director circle, describes a conic passing through the three vertices of the fixed triangle self-reciprocal with respect to the entire system of conics passing through the four points; and that (2) the circle circumscribing any triangle self-reciprocal with respect to any conic intersects at right angles the director circle of the conic.

N.B.—As the centre of a variable quadric surface passing through eight fixed points in space describes the partial intersection of two fixed quadrics, having a common generator, and passing through the four vertices of the fixed tetrahedron self-reciprocal with respect to the entire system of quadrics passing through the eight points; and as the sphere circumscribing any tetrahedron self-reciprocal with respect to any quadric surface intersects at right angles the director sphere of the quadric; the property in Geometry of Three Dimensions analogous to the above follows manifestly in precisely the same manner.

II. *Solution by* C. Leudesdorf, M.A.

The equation to any conic passing through the four points

$$lx = \pm my = \pm nz \text{ is } Lx^2 + My^2 + Nz^2 = 0,$$

with the condition $\dfrac{L}{l^2} + \dfrac{M}{m^2} + \dfrac{N}{n^2} = 0$ .....(1).

The director circle of the conic is (SALMON'S *Conics*, Art. 383)

$$L(M+N)x^2+\ldots+\ldots+2MN\cos A\,.\,yz+\ldots+\ldots=0,$$

or

$$\frac{y^2+z^2+2yz\cos A}{L}+\ldots+\ldots=0,$$

which, if r_1, r_2, r_3 be the distances of the point (xyz) from the vertices of the reference-triangle, is

$$r_1^2\frac{\sin^2 A}{L}+r_2^2\frac{\sin^2 B}{M}+r_3^2\frac{\sin^2 C}{N}=0.$$

The envelope of this, subject to the condition (1), is easily seen to be

$$r_1\frac{\sin A}{l}+r_2\frac{\sin B}{m}+r_3\frac{\sin C}{n}=0,$$

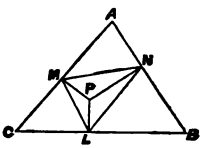
a bicircular quartic, three of whose foci are at the vertices of the reference-triangle, which is evidently the common self-conjugate triangle of the conics. [Another Solution is given on p. 23 of Vol. XXVII. of the *Reprint*.]

5175. (By the Rev. H. G. DAY, M.A.)—From a random point within the area of an acute-angled triangle, perpendiculars are drawn on the sides; show that the average area of the triangle formed by joining the feet of these perpendiculars is, in parts of the area of the given triangle,

$$\tfrac{1}{12}(\sin^2 A+\sin^2 B+\sin^2 C).$$

Solution by E. B. SEITZ; Rev. J. L. KITCHIN, M.A.; *and others.*

Let ABC be the given triangle, P the random point, PL, PM, PN the perpendiculars, and LMN the triangle formed. Put PL=x, PM=y, PN=z, BC=a, CA=b, AB=c, area ABC=Δ.



Then $$\text{area LMN}=\left(\frac{xy}{ab}+\frac{xz}{ac}+\frac{yz}{bc}\right)\Delta,$$

and $$ax+by+cz=2\Delta;$$

hence, substituting for z in terms of x and y, the average is

$$=\int_0^{\frac{2\Delta}{a}}\int_0^{\frac{1}{b}(2\Delta-ax)}\left(\frac{xy}{ab}+\frac{2x\Delta}{ac^2}-\frac{x^2}{c^2}-\frac{bxy}{ac^2}+\frac{2y\Delta}{bc^2}-\frac{axy}{bc^2}-\frac{y^2}{c^2}\right)dx\,\operatorname{cosec}C\,dy$$

$$=\int_0^{\frac{2\Delta}{a}}\left(\frac{4\Delta^3}{3b^3c^2}+\frac{4x\Delta^2\cos A}{ab^2c}-\frac{4x^2\Delta\cos A}{b^2c}-\frac{a^2x^2\Delta}{b^3c^2}+\frac{ax^3\cos A}{b^2c}+\frac{a^2x^3}{3b^3c^2}\right)\operatorname{cosec}C\,dx$$

$$=\tfrac{2}{3}\Delta^3\left(\frac{1}{b^2c^2}+\frac{\cos A}{a^2bc}\right)=\tfrac{1}{12}\Delta(\sin^2 A+\sin^2 B+\sin^2 C).$$

[The same result has been otherwise obtained by the EDITOR, in the Solution of his Question 3960, on p. 33 of Vol. XIX. of the *Reprint*, where it is further shown that, if P range over the circle drawn round the triangle ABC, the average area of LMN is one-eighth of the triangle ABC.]

5204. (By H. M. Dyer, B.A.)—If a semi-cubical parabola and a parabola have a common tangent at the vertices, and parallels to this tangent cut the semi-cubical in A′, B′, C′, and the parabola in A, B, C respectively; show that the ratio $\Delta A'B'C' : \Delta ABC$ varies as the sum of the products taken two at a time of the ordinates of A, B, C.

Solution by the Proposer; Prof. Evans, M.A.; *and others.*

Let $y^2 = cx^3$ and $y^2 = px$ be the two curves; and (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_1, y_1'), (x_2, y_2'), (x_3, y_3') the coordinates of A, B, C, A′, B′, C′;

then
$$\frac{\Delta A'B'C'}{\Delta ABC} = \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1' & y_2' & y_3' \end{vmatrix} \div \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

$$= \left(\frac{c}{p}\right)^{\frac{1}{2}} \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^{\frac{3}{2}} & x_2^{\frac{3}{2}} & x_3^{\frac{3}{2}} \end{vmatrix} \div \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^{\frac{1}{2}} & x_2^{\frac{1}{2}} & x_3^{\frac{1}{2}} \end{vmatrix}$$

$$= \left(\frac{c}{p}\right)^{\frac{1}{2}} \frac{\Sigma(x_2 x_3^{\frac{3}{2}} - x_3 x_2^{\frac{3}{2}})}{\Sigma(x_2 x_3^{\frac{1}{2}} - x_3 x_2^{\frac{1}{2}})}$$

$$= \left(\frac{c}{p}\right)^{\frac{1}{2}} \left\{(x_2 x_3)^{\frac{1}{2}} + (x_3 x_1)^{\frac{1}{2}} + (x_1 x_2)^{\frac{1}{2}}\right\}$$

$$= \left(\frac{c}{p^3}\right)^{\frac{1}{2}} (y_2 y_3 + y_3 y_1 + y_1 y_2).$$

5314. (By Alexander J. Ellis, F.R.S.)—Given $f(ax, y) = af(x, y)$ and $f(x, by) = bf(x, y)$, to find all the possible forms of $f(x, y)$. It is usually assumed that $f(x, y) = cxy$, where c is a constant, and this form evidently satisfies the conditions. But if $a = b$, then $f(x, y) = \frac{c}{xy}$ also satisfies them; and generally, if $a^m = a$, $b^n = b$, that is, if a is any one of the roots of a, and b any one of the roots of b, then $f(x, y) = x^m y^n$ also satisfies the conditions. The question becomes important in such problems as this:—The work done in carrying a load over a distance is multiplied when either the load or the distance is multiplied; what is the relation between work done, load, and distance?

Solution by Robert Rawson.

Let $cf(x, y) = f(ax, y)$, and $c_1 f(x, y) = f(x, by)$ .....(1), where c, c_1, a, b are given constants. If $c = a$ and $c_1 = b$, the equations (1) are those given in the question.

Put $f(x, y) = \phi(x, y)\,\psi(x)\,\psi_1(y)\, x^{\frac{\log c}{\log a}} y^{\frac{\log c_1}{\log b}}$ .....(2).

This function will satisfy equations (1) if

$$\phi(x,y) = \phi(ax,y) = \phi(x,by);\ \psi(x) = \psi(ax),\ \psi_1(y) = \psi_1(by) \ldots\ldots (3,4).$$

The general solution of equations (1) is reduced to the general solution of the more simple equations (3) and (4).

For a solution of (3) and (4) I may refer to De Morgan's *Calculus of Functions*, pp. 330, 374; to Boole's *Finite Differences*, p. 220; and to Professor Tanner's Solution of Quest. 5314 on pp. 58, 59 of this volume of the *Reprint*. Professor Tanner has correctly assumed $\phi(x,y)$ to be represented by $F(x) \cdot F_1(y)$; it may, however, be transformed into the following:—

$$\phi(x,y) = a(y)\{F(x) - F_1(y)\}\{F(x) - F_2(y)\} \ldots\ldots \{F(x) - F_n(y)\} \ldots\ldots (5),$$

where $F_1(y)$, $F_2(y) \ldots F_n(y)$ are the n roots of $\phi(x,y) = 0$ with respect to $F(x)$.

Equation (5) will evidently satisfy (3) if

$$a(y) = a(by),\ F(x) = F(ax),\ F_1(y) = F_1(by) \ldots\ldots F_n(y) = F_n(by) \ldots (6).$$

Hence the general solution of (1) is reduced to the general solution of

$$\beta(x) = \beta(hx) \ldots\ldots (7).$$

A solution of (7) is $$\beta(x) = A_m \cos\{2m\pi V(x) + B_m\} \ldots\ldots (8),$$

where (m) is any whole number and $V(x)$ satisfies $V(x) + 1 = V(hx)$. One solution of $V(x)$ is $\log(x)$ upon $\log(h)$.

With respect to the illustrative problem, I share the views of Professor Tanner. It may be that Mr. Ellis has not given sufficient weight to the fact that *work* is a *definition*, viz., the product of force and space. Any attempt, therefore, to infer its relation to force and space from two given equations must be exceedingly difficult.

Note on Professor Lloyd Tanner's "Note on the Solution of Question 5314."*

By Alexander J. Ellis, F.R.S.

1. If we limit log a to one of its values, then possibly the solution may be general so far. But there is no such limitation in the question.

2. I am not satisfied with the reasoning here, but am unable to go into the question at present. It however goes no way to prove the generality of the particular solution, but merely the extent to which it holds. The question made no limitation whatever, and my particular cases were merely illustrations.

3. I have not read Riemann, and do not precisely understand the ground of his reasoning, or the meaning of the words cited, as "längs einer Linie" and "überall" require considerable explanation. But I doubt any general conclusion based on particular cases. The first sentence in (3) sounds like a revival of the late Dean Peacock's "permanence of equivalent forms," which is generally discredited.

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* See pp. 59, 60 of this volume of the *Reprint*.

5169. (By J. L. McKenzie.)—Two parabolas have a common point O, and the diameter of each through O is a tangent to the other; a line drawn through O meets one parabola in Q and the other in R; and a point P is taken on the line so that $OP^2 = OQ \cdot OR$. Prove that the locus of P is an hyperbola having the diameters of the parabolas through O as asymptotes.

Solution by R. Tucker, M.A.; A. W. Cave; *and others.*

Let $y = mx$ be the equation to OP; and $y^2 = 4a'x \ldots(1)$, $x^2 = 4b'y \ldots(2)$, the two parabolas;

then, from (1), $4a'x = m^2x^2$, that is, $x = \frac{4a'}{m^2}$, $y = \frac{4a'}{m}$;

and, from (2), $x^2 = 4b'mx$, that is, $x = 4b'm$, $y = 4b'm^2$;

therefore $X^2Y^2 = xyx'y' = (16a'b')^2$, and $XY = 16a'b'$,

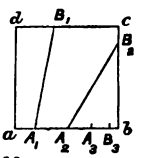
the equation to a hyperbola having the given diameters for asymptotes.

5313. (By Elizabeth Blackwood.)—If A, B, C, D are random points in the perimeter of a square; find the chance that the straight lines AB, CD will intersect within the square.

I. *Solution by* C. Leudesdorf, M.A.

Suppose A and B first taken at random in the perimeter. They may fall (1) on opposite sides of the square, (2) on adjacent sides, (3) on the same side; and the respective chances of these cases are $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$.

(1). If A and B fall on opposite sides, as A_1B_1. Then, in order that CD should intersect A_1B_1 within the square, the two points C, D must fall, one in A_1adB_1, and one in B_1cbA_1; so that if $A_1b = x$, $B_1c = y$, $ab = 1$, the chance in this case is



$$p_1 = 2\iint \frac{A_1adB_1 \times A_1bcB_1}{16}\,dx\,dy$$

$$= \frac{1}{8}\int_0^1\int_0^1 (1+x+y)(3-x-y)\,dx\,dy = \frac{23}{48}.$$

(2). If A and B fall on adjacent sides, as A_2, B_2: then, for favourable cases, the two points C, D must fall, one in A_2bB_2, one in A_2adcB_2: so that if $A_2b = x$, $bB_2 = y$, the chance in this case is

$$p_2 = 2\int\int \frac{A_2bB_2 \times A_2adcB_2}{16}\,dx\,dy$$

$$= \frac{1}{8}\int_0^1\int_0^1 (x+y)(4-x-y)\,dx\,dy = \frac{17}{48}.$$

(3). If A and B fall on the same side, as A_3B_3, then C and D may be distributed in 16 different ways on the perimeter. In eight of these AB and CD meet on the perimeter, in seven others they meet outside the square; in the other case the intersection is indeterminate (when A, B, C, D all fall on the same side). The eight favourable cases are the six where one of two, C or D, falls on the same side as AB, and the other on one of the other sides; and the two where C and D fall together on one of the sides adjacent to the side AB.

If then we consider points on the perimeter as lying within the square, we have for the chance in this case, $p_3 = \frac{1}{2}$.

The chance required is then

$$\tfrac{1}{4}p_1 + \tfrac{1}{2}p_2 + \tfrac{1}{4}p_3 = \frac{1}{4}\cdot\frac{23}{48} + \frac{1}{2}\cdot\frac{17}{48} + \frac{1}{4}\cdot\frac{1}{2} = \frac{27}{64}.$$

[After the foregoing solution had been set in type, the EDITOR pointed out to Mr. LEUDESDORF that he had not, in (1) and (2), included all the intersections that take place on the perimeter, inasmuch as such intersections will also occur when C, D are both *on the same side* of AB, provided that A, C, D or B, C, D are all three on the same side of the square. Thereupon, Mr. LEUDESDORF sent the following addition to his solution, which is thereby made exact in all respects:—]

(α). If we include perimeter intersections, we have not included all the cases in (1), as remarked by the EDITOR. For if C, D fall both in any one of the spaces A_1a, A_1b, B_1c, B_1d, we still have intersections on the perimeter. We must therefore add to the chance found $\frac{23}{48}$, the value of

$$\frac{1}{16}\int_0^1\int_0^1 dx\,dy\,\{x^2+(1-x)^2+y^2+(1-y)^2\}, \text{ that is, } \frac{1}{12}.$$

Doing this, we find for the correct value $\frac{9}{16}$. A similar remark applies to (2); where we have to add the value of

$$\frac{1}{16}\iint dx\,dy\,\{(A_2a)^2+(A_2b)^2+(B_2b)^2+(B_2c)^2\}, \text{ that is, } \frac{1}{12} \text{ again:}$$

thus we find for the chance in this case $\frac{17}{48}+\frac{1}{12}$ or $\frac{7}{16}$.

(β). But if we do *not* include perimeter intersections, then we have included too many cases in both (1) and (2). For if C and D fall both on *ab* or both on *cd* in (1), and both in *ab* or both in *bc* in (2), we have intersections on the perimeter. In each case we must therefore subtract from the chance found the value of

$$\frac{2}{16}\int_0^1\int_0^1 \{x(1-x)+y(1-y)\}\,dx\,dy, \text{ that is, } \frac{1}{24}.$$

Thus we find for the correct chances $\frac{7}{16}$, $\frac{5}{16}$.

(γ). The values found in (3) are correct: thus, on the hypothesis (α), we have $p_3 = \frac{5}{16}$; and on hypothesis (β), $p_3 = 0$.

If then P_1, P_2, P_3 be the chances of intersection (1) *inside*, (2) *on*, (3) *outside* the perimeter respectively, we have

$$P_1+P_2 = \frac{1}{4}\cdot\frac{9}{16}+\frac{1}{2}\cdot\frac{7}{16}+\frac{1}{4}\cdot\frac{8}{16} = \frac{31}{64},$$

$$P_1 = \frac{1}{4}\cdot\frac{7}{16}+\frac{1}{2}\cdot\frac{5}{16} = \frac{17}{64}; \text{ therefore } P_2 = \frac{14}{64}.$$

The chance of an indeterminate intersection being $\frac{1}{64}$, we find

$$P_3 = 1-P_1-P_2-\frac{1}{64} = \frac{32}{64}.$$

II. *Solution by* HUGH MCCOLL, B.A.

The chance will remain the same if we restrict one of the four random points, say A, to one side of the square, say the side *ab*. Let q = the re-

quired chance. Let p_1, p_2, p_3 = the respective chances that, in addition to the required intersection within the square, the point B will fall on *bc*, *cd*, or *da*. Then $q = p_1 + p_2 + p_3 = 2p_1 + p_2$.

Let x, y, z, u be the respective distances of the points A, B, C, D respectively from *b*, measured round the perimeter in the direction *badc*. Then, taking the side of the square as unity, we get

$$p_1 = 4\int_0^1 dx \int_3^4 dy \int_0^x dz \int_1^y du \div \int_0^1 dx \int_0^4 dy \int_0^4 dz \int_0^4 du = \frac{5}{64}.$$

Similarly we get

$$p_2 = \frac{1}{64}\left\{4\int_0^1 dx \int_2^3 dy \int_0^x dz \int_1^y du + 2\int_0^1 dx \int_2^3 dy \int_3^4 dz \int_x^y du\right\} = \frac{7}{64}.$$

Hence we have $$q = \frac{5}{32} + \frac{7}{64} = \frac{17}{64}.$$

[After receiving the foregoing investigation, the EDITOR requested Mr. MCCOLL, as he had previously requested Mr. LEUDESDORF, to calculate the respective chances of intersections (1) *inside*, (2) *on*, (3) *outside* the perimeter; whereupon Mr. MCCOLL sent the following addition to his solution which now, except in a slight matter of form, agrees with Mr. LEUDESDORF's results :—]

Let p_1 = the chance of an intersection *inside the square*,
p_2 = the chance of an intersection *on the perimeter*,
p_3 = the chance of an intersection *outside the square*,
p_4 = the chance that AB and CD *will be parallel*,
p_5 = the chance that AB and CD *will be in the same straight line*.

Now Mr. LEUDESDORF and I have both found, and found correctly, that $p_1 = \frac{17}{64}$. It therefore remains to find the other chances, and this we may do without integration as follows:—

Let the sides of the square be marked 1, 2, 3, 4. As before observed, the chances remain the same if we restrict one of the points, say A, to one side of the square, say the side 1. Let A then be so restricted, while B, C, D may fall on any of the four sides. I will now use the principles and notation of my *Symbolical Language*. [See pp. 20—23, 78, 79 of this volume of the *Reprint*.]

Let B_1, B_2, B_3, B_4 denote the respective statements that B will fall on the side 1, on the side 2, on the side 3, or on the side 4; and let C_1, C_2, &c., D_1, D_2, &c., be similarly interpreted.

Let Q denote the following statement:—AB and CD will neither intersect inside nor outside the square; and let q = the chance that the statement Q is true. Then $q = p_2 + p_4 + p_5 = 1 - p_1 - p_3$.

Now $Q = Q(B_1 + B_2 + B_3 + B_4)(C_1 + C_2 + C_3 + C_4)(D_1 + D_2 + D_3 + D_4)$, for the product of the three indeterminate factors = 1. Each of the 64 terms of which this product consists is a statement, the chance of whose truth (since A is restricted to the side 1) is $\frac{1}{64}$. Of these 64 terms 16 are consistent with, and in fact imply, the outside factor Q, while the remaining 48 terms are inconsistent with Q.

Hence we have $q = \frac{16}{64}$, and $q = 1 - p_1 - p_3$;
therefore $p_3 = 1 - p_1 - q = 1 - \frac{17}{64} - \frac{16}{64} = \frac{31}{64}$.
Again $p_4 = (\frac{1}{4})^2 = \frac{1}{16}$, and $p_5 = (\frac{1}{4})^2 = \frac{1}{16}$;
therefore $p_2 = 1 - \frac{17}{64} - \frac{31}{64} - \frac{1}{64} - \frac{1}{64} = \frac{14}{64}$.
Hence, finally, we obtain $p_1, p_2, p_3, p_4, p_5 = \frac{17}{64}, \frac{14}{64}, \frac{31}{64}, \frac{1}{64}, \frac{1}{64}$.

[Miss BLACKWOOD, the Proposer, considers that she has obtained an ap-

proximate solution of her problem by *experiment*, and sends the following as the result she has thereby arrived at:—

She first made 15 trials, whereof "14 times running there were no intersections inside the square, and the last time there was;" and, again, on a second attempt, "the event whose chance is required happened 57 times in 200 trials. If to these the former 15 trials be added, the chance, as thus determined by experiment, is $\frac{57}{200}\frac{5}{15}$."

Whatever may be the value of Miss BLACKWOOD's experiments, the close agreement between her result and the exact value is very remarkable; for we have Experimental value $= \frac{57}{200}\frac{5}{15} = \cdot 269\ldots$,

Exact value $= \frac{11}{64} = \cdot 265\ldots]$.

5211. (By Professor TOWNSEND, F.R.S.)—A circular cylinder, whose centre of inertia is not on its axis of figure, being supposed to roll without sliding on a rough horizontal plane, and to have the velocity of rotation just sufficient to carry it with exhausted energy to its position of unstable equilibrium against the action of gravity; determine, in finite terms, its time of passage from the opposite position of stable equilibrium to any other on its way.

I. *Solution by* J. W. SHARPE, B.A.

Let a = radius of cylinder; b = distance of centre of inertia from the axis of figure; θ = angle which the perpendicular from the centre of inertia to the axis makes with the *downward* vertical from the axis; k^2 = square of radius of gyration of cylinder about the principal axis at its centre of inertia parallel to the axis of figure. Then the equation of energy gives clearly, by the condition of the question,

$$(k^2+a^2+b^2-2ab\cos\theta)\left(\frac{d\theta}{dt}\right)^2 = 2gb(1+\cos\theta);$$

therefore
$$(2gb)^{\frac{1}{2}}t = \int_0^{\theta} \frac{(k^2+a^2+b^2-2ab\cos\theta)^{\frac{1}{2}}}{(1+\cos\theta)^{\frac{1}{2}}}\,d\theta,$$

therefore
$$t = 2\left(\frac{a}{g}\right)^{\frac{1}{2}}\int_0^{x} \frac{(a^2+x^2)^{\frac{1}{2}}}{1-x^2}\,dx;$$

where $x = \sin\frac{1}{2}\theta$, $a^2 = \frac{k^2+(a-b)^2}{4ab}$;

or
$$t = 2\left(\frac{a}{g}\right)^{\frac{1}{2}}\int_0^{x}\left\{\frac{a^2+1}{(1-x^2)(a^2+x^2)^{\frac{1}{2}}} - \frac{1}{(a^2+x^2)^{\frac{1}{2}}}\right\}dx.$$

The 2nd and 1st terms in the integral give, respectively, if $u = 1+x$,

$$2\left(\frac{a}{g}\right)^{\frac{1}{2}}\log\frac{a}{(a^2+x^2)^{\frac{1}{2}}+x},$$

$$\tfrac{1}{2}\int_0^{x}\frac{dx}{(1+x)(a^2+x^2)^{\frac{1}{2}}} + \tfrac{1}{2}\int_0^{x}\frac{dx}{(1-x)(a^2+x^2)^{\frac{1}{2}}} = \tfrac{1}{2}\int_{2-u}^{u}\frac{du}{u\{a^2+(u-1)^2\}^{\frac{1}{2}}}.$$

Putting $u = y^{-1}$, and integrating, we get for the integral

$$\frac{1}{(a^2+1)^{\frac{1}{2}}} \log \frac{u}{2-u} \frac{(a^2+1)^{\frac{1}{2}} \left\{a^2+(u-1)^2\right\}^{\frac{1}{2}} + a^2 + u - 1}{(a^2+1)^{\frac{1}{2}} \left\{a^2+(u-1)^2\right\}^{\frac{1}{2}} + a^2 + 1 - u}.$$

If we put r^2 for $a^2 + b^2 - 2ab\cos\theta$, that is, if r denote the distance of the centre of inertia from the lowest generating line of the cylinder, at the time t, where t denotes the interval since the time the centre of inertia passed through the position of stable equilibrium, the value of t is

$$2\left(\frac{a}{g}\right)^{\frac{1}{2}} \log \frac{\left\{k^2+(a-b)^2\right\}^{\frac{1}{2}}}{(k^2+r^2)^{\frac{1}{2}} + 2(ab)^{\frac{1}{2}} \sin\frac{1}{2}\theta} + \left(\frac{b}{g}\right) \frac{2a}{\left\{k^2+(a+b)^2\right\}^{\frac{1}{2}}} \log \frac{1+\sin\frac{1}{2}\theta}{1-\sin\frac{1}{2}\theta}$$
$$\times \frac{\left\{k^2+(a+b)^2\right\}^{\frac{1}{2}} (k^2+r^2)^{\frac{1}{2}} + k^2 + (a-b)^2 + 4ab\sin\frac{1}{2}\theta}{\left\{k^2+(a+b)^2\right\}^{\frac{1}{2}} (k^2+r^2)^{\frac{1}{2}} + k^2 + (a-b)^2 - 4ab\sin\frac{1}{2}\theta}.$$

This gives $t=0$ when $\theta=0$, and $t=\infty$ when $\theta=\pi$, as it should do.

II. *Solution by the* Proposer.

Denoting by a the radius of the cylinder, by b the distance OL of its centre of inertia O from its axis of figure L, by k its arm of inertia with respect to the axis through O parallel to L, by X its line of contact with the plane in any position, by θ the angle OLX corresponding to the position, by ω the velocity of rotation $\frac{d\theta}{dt}$ corresponding to the same, and by θ' and ω' the values of θ and ω corresponding to any other position of the body; then, since generally, by the equation of *vis viva*,

$$(a^2+b^2-2ab\cos\theta+k^2)\,\omega^2 - (a^2+b^2-2ab\cos\theta'+k^2)\,\omega'^2 = 2gb(\cos\theta-\cos\theta'),$$

therefore when, as by hypothesis, $\omega'=0$ when $\cos\theta'=-1$, then

$$(a^2+b^2-2ab\cos\theta+k^2)\,\omega^2 = 2gb(1+\cos\theta),$$

which, by an obvious transformation, gives

$$(gb)^{\frac{1}{2}} \cdot dt = \left\{\left[(a-b)^2+k^2\right] + \left[(a+b)^2+k^2\right]\tan^2\tfrac{1}{2}\theta\right\}^{\frac{1}{2}} \cdot d\tfrac{1}{2}\theta,$$

both sides of which are exact differentials; hence, by integration in finite terms, supposing θ and t to commence together, we get

$$(gb)^{\frac{1}{2}} \cdot t = \left[(a+b)^2+k^2\right]^{\frac{1}{2}}$$
$$\times \log \frac{\left[(a+b)^2+k^2\right]^{\frac{1}{2}} \cdot \tan\frac{1}{2}\theta + \left\{\left[(a-b)^2+k^2\right] + \left[(a+b)^2+k^2\right]\tan^2\tfrac{1}{2}\theta\right\}^{\frac{1}{2}}}{\left[(a-b)^2+k^2\right]^{\frac{1}{2}}}$$
$$-2(ab)^{\frac{1}{2}} \cdot \log \frac{2(ab)^{\frac{1}{2}} \cdot \sin\frac{1}{2}\theta + \left\{\left[(a-b)^2+k^2\right] + 4ab \cdot \sin^2\frac{1}{2}\theta\right\}^{\frac{1}{2}}}{\left[(a-b)^2+k^2\right]^{\frac{1}{2}}},$$

which accordingly is the general formula, in finite terms, for t in terms of θ.

Putting $\theta=\pi$ in the above formula, we get $t=\infty$; which accords with the general property that, under circumstances similar to the above, a position of unstable equilibrium can be reached with exhausted energy, by a body or system of bodies in motion, only at the expiration of an infinite time.

5325. (By S. Tebay, B.A.)—Show that the whole attractive force on an undisturbed planet varies as the angle swept out by the radius vector; and thence show that there can be no accumulated resultant force in any direction when the planet has performed a complete revolution.

I. *Solution by* Professor Lloyd Tanner, M.A.

Taking the "whole attractive force" to mean the time-integral of the force, it is $= \int_{t_1}^{t_2} \frac{\mu}{r^2}\, dt = \int_{\theta_1}^{\theta_2} \frac{\mu}{r^2} \cdot \frac{dt}{d\theta}\, d\theta = \int_{\theta_1}^{\theta_2} \frac{\mu}{h}\, d\theta = \frac{\mu(\theta_2 - \theta_1)}{h}$,

where the constant h is twice the area described by radius vector in a unit of time.

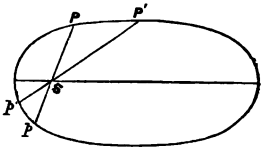
II. *Solution by the* Proposer.

Let $r\theta$ be the position of the planet at the time t, and let $F = \frac{\mu}{r^2}$. We may suppose F to be constant during the element of time dt.

Hence $\int F\, dt = \int \frac{\mu}{r^2}\, dt.$

But $r^2 \frac{d\theta}{dt} = h$; therefore

$$\int F\, dt = \frac{\mu}{h} \int d\theta = \frac{\mu}{h}\theta + c.$$



Let P, P′ be any two positions of the planet, S the Sun, and $\angle PSP' = \alpha$. Then the whole force impressed on the planet in passing from P to P′ $= \frac{\mu}{h}\alpha$; and the resultant force will evidently bisect the angle PSP′.

Let PS, P′S be produced to meet the ellipse in p, p'; then the whole force on the planet while it passes from p to p' is also $\frac{\mu}{h}\alpha$, and the resultant bisects the angle pSp'. This will therefore neutralise the former resultant. Hence there can be no accumulated resultant in any direction when the planet has performed a complete revolution.

5429. (By Professor Lloyd Tanner, M.A.)—If

$$\frac{d^2z}{dx^2} \cdot \frac{d^2z}{dy^2} - \left(\frac{d^2z}{dx\,dy}\right)^2 = 0;$$

then, x being considered as a function of z, y,

$$\frac{d^2x}{dz^2} \cdot \frac{d^2x}{dy^2} - \left(\frac{d^2x}{dz\,dy}\right)^2 = 0.$$

I. *Solution by* J. J. WALKER, M.A.

More generally, on the supposition of the Question, we have

$$\frac{d^2x}{dz^2}\frac{d^2x}{dy^2}-\left(\frac{d^2x}{dz\,dy}\right)^2=\left(\frac{dx}{dz}\right)^4\left\{\frac{d^2z}{dx^2}\frac{d^2z}{dy^2}-\left(\frac{d^2z}{dx\,dy}\right)^2\right\}.$$

For $\frac{d^2x}{dz^2}=-\left(\frac{dx}{dz}\right)^3\frac{d^2}{dx^2}$ .....(1); also $\frac{dx}{dy}=-\frac{dx}{dz}\frac{dz}{dy}$,

whence $\frac{d^2x}{dy^2}=-\frac{d^2x}{dz\,dy}\frac{dz}{dy}-\frac{dx}{dz}\left(\frac{d^2z}{dy^2}+\frac{d^2z}{dx\,dy}\frac{dx}{dy}\right.$ or $\left.-\frac{dx}{dz}\frac{d^2z}{dx\,dy}\frac{dz}{dy}\right)$...(2).

From (1) and (2), we obtain

$$\frac{d^2x}{dz^2}\frac{d^2x}{dy^2}=\left(\frac{dx}{dz}\right)^4\frac{d^2z}{dx^2}\frac{d^2z}{dy^2}-\left\{\frac{d^2x}{dz\,dy}-\left(\frac{dx}{dz}\right)^2\frac{d^2z}{dx\,dy}\right\}\frac{d^2x}{dz^2}\frac{dz}{dy}\quad\text{.....(3).}$$

From $\frac{dz}{dx}=\left(\frac{dx}{dz}\right)^{-1}$, we have $\left(\frac{dx}{dz}\right)^2\frac{d^2z}{dx\,dy}=-\frac{d^2x}{dz\,dy}-\frac{d^2x}{dz^2}\frac{dz}{dy}$,

whence $\left(\frac{d^2x}{dz\,dy}\right)^2=\left(\frac{dx}{dz}\right)^4\left(\frac{d^2z}{dx\,dy}\right)^2-\left\{\frac{d^2x}{dz\,dy}-\left(\frac{dx}{dz}\right)^2\frac{d^2z}{dx\,dy}\right\}\frac{d^2x}{dz^2}\frac{dz}{dy}$

[for, if $p=-q-r$, then $q^2=p^2-(p-q)r$]. Subtracting (3) from this last equality, the generalization above given follows.

II. *Solution by the* PROPOSER.

In forming the equation $\frac{d^2z}{dx^2}\cdot\frac{d^2z}{dy^2}-\left(\frac{d^2z}{dx\,dy}\right)^2=0$ .....(1),

the variable z is not dealt with like x, y. The want of symmetry of the equation is due to this fact, as we can show by forming equivalent equations in which z, x, y are treated alike; for in such equations it will be found that x, y, z are similarly involved. For example, take any solution of (1), say $f(x, y, z)=0$, and form the equation equivalent to (1) with f as dependent and x, y, z as independent variables. This equation is

$$\begin{vmatrix} & \frac{df}{dx}, & \frac{df}{dy}, & \frac{df}{dz} \\ \frac{df}{dx}, & \frac{d^2f}{dx^2}, & \frac{d^2f}{dx\,dy}, & \frac{d^2f}{dx\,dz} \\ \frac{df}{dy}, & \frac{d^2f}{dx\,dy}, & \frac{d^2f}{dy^2}, & \frac{d^2f}{dy\,dz} \\ \frac{df}{dz}, & \frac{d^2f}{dx\,dz}, & \frac{d^2f}{dy\,dz}, & \frac{d^2f}{dz^2} \end{vmatrix}=0.$$

Now this form is not affected by an interchange of any pair of the three variables x, y, z; and the same must be true of its equivalent (1).

The same follows from the solution of (1), which may be written

$$\left.\begin{aligned} x\phi(u)+y\psi(u)+z\chi(u)+\theta(u)&=0 \\ x\phi'(u)+y\psi'(u)+z\chi'(u)+\theta'(u)&=0 \end{aligned}\right\}.$$

III. *Solution by* Robert Rawson.

The first equation implies that z is a function of x, y.

If we consider that y is independent and not a function of x, as is sometimes the case in the solution of partial differential equations, then we may put $z = \mathrm{R}$, a function of x, y .....(1).

From (1) it readily follows by differentiation that

$$\frac{d^2z}{dx^2}\cdot\frac{d^2z}{dy^2}-\left(\frac{d^2z}{dx\,dy}\right)^2=\frac{d^2\mathrm{R}}{dx^2}\cdot\frac{d^2\mathrm{R}}{dy^2}-\left(\frac{d^2\mathrm{R}}{dx\,dy}\right)^2 \quad\ldots\ldots\ldots (2).$$

Again, differentiate (1), regarding x as a function of z, y; then

$$-\frac{d\mathrm{R}}{dx}\cdot\frac{d^2x}{dz^2}=\frac{d^2\mathrm{R}}{dx^2}\left(\frac{dx}{dz}\right)^2 \quad\ldots\ldots\ldots (3),$$

$$-\frac{d\mathrm{R}}{dx}\cdot\frac{d^2x}{dy^2}=\frac{d^2\mathrm{R}}{dx^2}\left(\frac{dx}{dy}\right)^2+2\left(\frac{d^2\mathrm{R}}{dx\,dy}\right)\frac{dx}{dy}+\frac{d^2\mathrm{R}}{dy^2} \quad\ldots\ldots\ldots (4),$$

$$-\frac{d\mathrm{R}}{dx}\cdot\frac{d^2x}{dy\,dz}=\frac{d^2\mathrm{R}}{dx^2}\cdot\frac{dx}{dz}\cdot\frac{dx}{dy}+\frac{d^2\mathrm{R}}{dx\,dy}\cdot\frac{dx}{dz} \quad\ldots\ldots\ldots (5).$$

From (3), (4), and (5) it follows that

$$\left(\frac{d\mathrm{R}}{dx}\right)^2\left\{\frac{d^2x}{dz^2}\cdot\frac{d^2x}{dy^2}-\left(\frac{d^2x}{dy\,dz}\right)^2\right\}=\left(\frac{dx}{dz}\right)^2\left\{\frac{d^2\mathrm{R}}{dx^2}\cdot\frac{d^2\mathrm{R}}{dy^2}-\left(\frac{d^2\mathrm{R}}{dx\,dy}\right)^2\right\} \quad..(6).$$

From (6) the truth of the proposition is obvious.

This very interesting property is useful in suggesting the form of the primitive (1). This form must be sought for in $\phi(x, y, z) = 0$, an equation which must give, by solving for x and z respectively, $x = f(y, z)$ and $z = f(y, x)$.

From this consideration (1) becomes

$$z=\frac{ay^2+by+\frac{b^2}{4a}}{x} \quad\ldots\ldots\ldots (7),$$

where a, b are arbitrary constants.

The equation (7) may, I presume, be called the *complete* primitive; the *general* primitive, if there is one, is not readily detected.

IV. *Solution by* J. W. Sharpe, B.A.

We have to transform the equation $rt-s^2=0$, the usual notation for partial differentials being assumed. Let us assume the solution of this equation under the form $x-\phi(yz)=0$, where $\phi(yz)$ does not contain x.

Differentiating with respect to y and z on the understanding that x is still the dependent variable, we obtain the following five equations:—

$$1-\phi_z p=0,\quad \phi_y+\phi_z q=0,\quad \phi_{yz}p+\phi_{zz}pq+\phi_z s=0,\quad \phi_{zz}p^2+\phi_z r=0,$$
$$\phi_{yy}+2\phi_{yz}q+\phi_{zz}q^2+\phi_z t=0.$$

From these five equations, combined with $rt-s^2=0$, we proceed to eliminate the five partial differentials p, q, r, s, t. We obtain

$$p=\frac{1}{\phi_z},\quad q=-\frac{\phi_y}{\phi_z};$$

therefore $\quad -s(\phi_z)^2 = \phi_z\phi_{yz} - \phi_y\phi_{zz}, \quad -r(\phi_z)^2 = \phi_{zz},$

and $\quad -t(\phi_z)^2 = (\phi_y)^2\phi_{zz} - 2\phi_y\phi\ \phi_{yz} + (\phi_z)^2\phi_{yy}.$

Hence, substituting in the last of the six equations, we have

$$\{(\phi_y)^2\phi_{zz} - 2\phi_y\phi_z\phi_{yz} + (\phi_z)^2\phi_{yz}\}\phi_{zz} - \{(\phi_z)^2(\phi_{yz})^2 + (\phi_y)^2(\phi_{zz})^2 - 2\phi_y\phi_z\phi_{yz}\phi_{zz}\} = 0,$$

or $$\phi_{yy}\phi_{zz} - (\phi_{yz})^2 = 0.$$

But, on the supposition that x is the dependent variable,

$$\phi_{yy} \equiv \frac{d^2x}{dy^2}, \quad \phi_{zz} \equiv \frac{d^2x}{dz^2}, \quad \phi_{yz} \equiv \frac{d^2x}{dy\,dz};$$

therefore $$\frac{d^2x}{dy^2}\cdot\frac{d^2x}{dz^2} - \left(\frac{d^2x}{dy\,dz}\right)^2 = 0$$

is the required transformation.

The above method of transformation occasionally proves convenient, especially when the given equation is linear in r, s, t.

For example, take the equation

$$q(1+q)r - (p+q+2pq)s + p(1+p)t = 0 \quad \text{.....(A).}$$

Let us change the dependent variable to y.

If we multiply over three equations of transformation in r, s, t, by the coefficients of r, s, t respectively, and add, we get at once, by virtue of (A),

$$(q+1)\phi_{xx} + (p-q)\phi_{xz} = 0, \quad \text{or} \quad \left(1+\frac{dy}{dz}\right)\frac{d^2y}{dx^2} - \left(1+\frac{dy}{dx}\right)\frac{d^2y}{dz\,dx} = 0,$$

which is easily solved.

5412. (By R. W. Genese, M.A.)—Prove that the product of the two normals that can be drawn to a parabola from a point on it is equal to 2SP . PG.

I. *Solution by* H. L. Richard, M.A.; Prof. Evans, M.A., *and others.*

Product of normals : normal chord through (x, y) = square of ordinate : parameter ($= x : 1$);

that is, $$N_1N_2 : \frac{2}{y^2}(y^2+aL)^{\frac{3}{2}} = y^2 : L \quad (\text{where } L = 4a);$$

$$\therefore \quad N_1N_2 = \frac{2}{L}(y^2+aL)^{\frac{3}{2}} = 2\left(\frac{y^2}{L}+a\right)(y^2+aL)^{\frac{1}{2}} = 2SP\,.\,PG.$$

II. *Solution by* R. F. Davis, B.A.; Christine Ladd; *and others.*

Let Q, R be the two points the normals at each of which pass through the point P(x, y) on the curve. Then, if T be the intersection of the tangents at Q, R, the coordinates of T are $(2a, -\frac{1}{2}y)$, and the equation to

QR is $Yy + 4a(X + 2a) = 0$ (as formerly proved in the *Reprint*). But TP is the diameter of the circle circumscribing PQR; hence, if PL be perpendicular to QR, $PQ \cdot PR = PT \cdot PL$. Now PT^2 is found to $= (x+a)(x+4a)$; and $PL^2 = 16a(x+a)^2(x+4a)^{-1}$, &c.

III. *Solution by* J. W. Sharpe, B.A.; J. O'Regan; *and others.*

Take the equation to the parabola under the form $x = au^2$, $y = 2au$, u being a variable parameter.

Let P be the point (u); Q, Q′ points (v), (v'), where QP, Q′P are normals at Q, Q′ respectively.

The equation to the normal at Q is $vx + y - av^3 - 2av = 0$; and this passes through the point (v); therefore, dividing out the factor $u - v$, we find that v, v' are the roots of the quadratic $z^2 + uz + 2 = 0$.

Now $$SP \cdot PG : a^2 = \{(u^2-1)^2 + 4u^2\}^{\frac{1}{2}} \{4 + 4u^2\}^{\frac{1}{2}} = 2\{1+u^2\}^{\frac{1}{2}},$$

and $$PQ \cdot PQ' : a^2 = 2(2u^2+2)\{1+u^2\}^{\frac{1}{2}} = 4\{1+u^2\}^{\frac{1}{2}};$$

therefore $$PQ \cdot PQ' = 2SP \cdot PG.$$

5013. (By Professor Townsend, F.R.S.)—The several particles m, m', &c., of a material system moving freely in space, being supposed to come in simultaneous collision each with every other particle of the system, shew that the entire loss of *vis viva* consequent on their collision

$$= \frac{1-e^2}{\Sigma(m)} \cdot \Sigma\{mm'[(u-u')^2 + (v-v')^2 + (w-w')^2]\};$$

where u, v, w, u', &c. are the components of the velocities of m, m', &c., before collision, and e the coefficient of elasticity supposed the same for all the particles of the system.

Solution by the Proposer.

Since, on elementary dynamical principles,

$$\Sigma m(u-u_1) = 0, \quad \Sigma m(v-v_1) = 0, \quad \Sigma m(w-w_1) = 0,$$

where u_1, v_1, w_1, &c., are the changed values of u, v, w, &c., consequent on the collision; therefore, by a well known consequence,

$$\Sigma(m) \cdot \Sigma\{m[(u-u_1)^2 + \&c.]\} = \Sigma\{mm'[\{(u-u_1)-(u'-u_1')\}^2 + \&c.]\}$$

$$= \Sigma\{mm'[\{(u-u')-(u_1-u_1')\}^2 + \&c.]\} = (1+e)^2 \cdot \Sigma mm'[(u-u')^2 + \&c.];$$

but, by Carnot's Theorem (see Poisson's *Traité de Mécanique*, 2nd edit., Art. 572), the entire loss of *vis viva* consequent on the collision

$$= \frac{1-e}{1+e} \cdot \Sigma\{m[(u-u_1)^2 + (v-v_1)^2 + (w-w_1)^2]\};$$

and therefore, &c.

5311. (By the EDITOR.)—If, in a triangle ABC, P be the orthocentre, O_1, O_2, O_3 the centres of the escribed circles, and Q the centre of the circumscribed circle; prove that

$$\begin{vmatrix} 1, & 1, & 1 \\ \overline{PO_1}^2, & \overline{PO_2}^2, & \overline{PO_3}^2 \\ \overline{QO_1}^2, & \overline{QO_2}^2, & \overline{QO_3}^2 \end{vmatrix} = \frac{a^2b^2c^2}{r^2}\left(\frac{b-c}{a}+\frac{c-a}{b}+\frac{a-b}{c}\right).$$

Solution by the PROPOSER.

In the Solutions of the EDITOR's Questions 5222, 5312 (*Reprint*, Vol. XXVII., pp. 85—87, and Vol. XXVIII., p. 24), and in Dr. BOOTH's *New Geometrical Methods*, Vol. II., p. 299, eq. (a), it is shown that

$$\overline{PO_1}^2 = 2r_1^2 - 2R\rho, \quad \overline{QO_1}^2 = R^2 + 2Rr_1;$$
$$\overline{PO_2}^2 = 2r_2^2 - 2R\rho, \quad \overline{QO_2}^2 = R^2 + 2Rr_2;$$

hence $$\begin{vmatrix} \overline{PO_1}^2, & \overline{PO_2}^2 \\ \overline{QO_1}^2, & \overline{QO_2}^2 \end{vmatrix} = 2R^2(r_1^2 - r_2^2) + 4Rr_1r_2(r_1 - r_2) + 4R^2\rho(r_1 - r_2).$$

Finding the two other similar values, we obtain for their sum—which is the determinant in the Question—the equivalent expressions

$$4R\left\{r_1^2(r_2-r_3)+r_2^2(r_3-r_1)+r_3^2(r_1-r_2)\right\},$$
$$\frac{4Rs^2}{\Delta}\left\{s_1(s_2^2-s_3^2)+s_2(s_3^3-s_1^2)+s_3(s_1^2-s_2^2)\right\},$$
$$\frac{4Rs^2}{\Delta}\left\{as_1(c-b)+bs_2(a-c)+cs_3(b-a)\right\},$$
$$\frac{4Rs^2}{\Delta}\left\{bc(s_3-s_2)+ca(s_1-s_3)+ab(s_2-s_1)\right\},$$
$$\frac{4Rs}{r}\left\{bc(b-c)+ca(c-a)+ab(a-b)\right\},$$
$$\frac{2R}{r}\left\{bc(b^2-c^2)+ca(c^2-a^2)+ab(a^2-b^2)\right\},$$
$$\frac{2abcR}{r}\left(\frac{b^2-c^2}{a}+\frac{c^2-a^2}{b}+\frac{a^2-b^2}{c}\right), \quad \frac{a^2b^2c^2}{r^2}\left(\frac{b-c}{a}+\frac{c-a}{b}+\frac{a-b}{c}\right).$$

5413. (By the Rev. A. F. TORRY, M.A.)—Two particles describe the same ellipse subject to the same force in the centre: show that their directions of motion at any time intersect on a similar ellipse.

I. *Solution by* J. W. SHARPE, M.A.; A. COHEN, M.A.; *and others.*

The radius vector of the particle sweeps out equal areas in equal times. Consider the orthogonal circular projection. The projected radius vector sweeps out equal areas about the centre of the circle in equal times; hence the projected particle has uniform motion; hence the lines of motion of

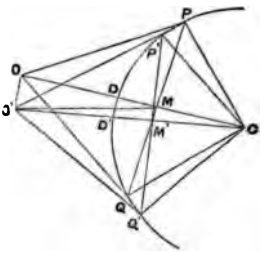
the projected particles intersect on a concentric circle; therefore the actual lines of motion intersect on a similar and similarly situated concentric ellipse.

II. *Solution by* C. BICKERDIKE; R. F. DAVIS, B.A.; *and others.*

Let p, q be corresponding points on the auxiliary circle; then p, q have the same constant velocity, so that the chord pq is of constant length. The tangents at p, q will therefore intersect on a fixed concentric circle, and those at P, Q on a fixed similar, similarly situated, and concentric conic.

III. *Solution by* J. J. WALKER, M.A.; Q. W. KING; *and others.*

Let C be the centre of the ellipse; P, Q and P′, Q′ simultaneous positions of the two particles at the beginning and end of an indefinitely short interval of time; PO, OQ and P′O′, O′Q′ tangents at the points P, Q, P′, Q′. Since the areas PCP′, QCQ′ are equal, the difference of the areas PCQO, P′CQ′O = that of POO′, QOO′. But the same difference = 2(CPO − CP′O′) = 2(PCP′ + POO′ − COO′); hence COO′ [½(POO′ + QOO′) or] MOO′ = PCP′; *i.e.*, CMO′ = PCP′; and similarly CM′O = QCC′, whence OO′ is parallel to MM′, *i.e.*, ultimately, to the tangent at D.



Again, COO′ : CMO′ = CO : CM = CO² : CD² = COO′ : CDD′. Hence CDD′ = CPP′; and while a particle described DD′ subject to the given force to the centre, O would describe the corresponding arc OO′ of a similar, concentric, and similarly placed ellipse.

5293. (By R. E. RILEY, B.A.)—Chords of constant length ($2c$) are drawn in an ellipse; prove that the locus of their poles with regard to the ellipse is the curve

$$b^4(b^2-c^2)x^4 + a^4(a^2-c^2)y^4 + a^2b^2(a^2+b^2-2c^2)x^2y^2 = a^2b^2(a^4y^2+b^4x^2).$$

Solution by the EDITOR.

In the EDITOR's Solution of Quest. 4519 (*Reprint*, Vol. XXII., p. 77), and afterwards in Dr. BOOTH's *New Geometrical Methods* (Vol. II., p. 425, Art. 348), it is shown, by the method of Tangential Coordinates, that the equation to the envelop of equal chords ($2c$) of a given ellipse $a^2y^2 + b^2x^2 = a^2b^2$, is $a^2b^2(\xi^2+v^2)(a^2\xi^2+b^2v^2-1) = c^2(a^2\xi^2+b^2v^2)^2$ ..... (1).

Now as in Conics the projective coordinates (x, y) of a pole, and the tangential coordinates (ξ, v) of its polar, are connected by the simple relations $a^2\xi = x$, $b^2v = y$, we have only in the equation (1) to substitute these values of ξ and v, and we shall obtain for the projective equation of the pole the result given in the question.

4836. (By Professor WOLSTENHOLME, M.A.)—Three conics S_1, S_2, S_3 are such that the polar reciprocal of any one with respect to another is the third; a triangle ABC is inscribed in S_1 and its sides touch S_2 in a, b, c; prove that the triangle abc is self-conjugate to S_1, and that its sides touch S_3; also the tangents to S_1 at A, B, C form a triangle A'B'C' which is self-conjugate to S_2 and whose angular points lie on S_3; also the tangent at A' passes through b, c, &c.

Solution by the PROPOSER.

Taking the triangle ABC as that of reference, let the equation of S_1 be $ayz+bzx+cxy=0$, and of S_2 $l^2x^2+\ldots-2mnyz-\ldots=0$; then the polar reciprocals of S_2, S_1 with respect to S_1, S_2 respectively, will be

$$(lbcx^2+mcay^2+nabz^2)+(ayz+bzx+cxy)(al+bm+cn)=0,$$

$$4lmn(bclx^2+camy^2+abnz^2)-(al+bm+cn)^2(l^2x^2+m^2y^2+n^2z^2)$$
$$+2(al+bm+cn)[(bm+cn-al)mnyz+\ldots+\ldots]=0;$$

whence, if $al+bm+cn=0$, each of them reduces to $bclx^2+camy^2+abnz^2=0$. Under this condition therefore the three conics form a harmonic system, and it is easily found that the invariants Θ, Θ' vanish for any two of the three. The form of the equation of S_3 proves that the triangle ABC is self-conjugate to S_3.

The sides of the triangle abc are $-lx+my+nz=0$, &c., and the condition that these may touch S_3 is $\frac{l^2}{lbc}+\ldots=0$, or $al+bm+cn=0$. Also these will form a triangle self-conjugate to S_1, if S_1 can be put in the form $A(-lx+my+nz)^2+B(lx-my+nz)^2+C(lx+my-nz)^2=0$, whence we must have

$$A+B+C=0, \quad \frac{(B+C-A)mn}{a}=\frac{(C+A-B)nl}{b}=\frac{(A+B-C)lm}{c},$$

or

$$\frac{A}{al}=\frac{B}{bm}=\frac{C}{cn},$$

with the condition $al+bm+cn=0$.

The tangents to S_1 at A, B, C are $cy+bz=0$, &c., and these will form a triangle self-conjugate to S_2 if S_2 can be put in the form

$$A'(cy+bz)^2+B'(az+cx)^2+C'(bx+ay)^2=0,$$

whence

$$\frac{A'bc}{mn}=\frac{B'ca}{nl}=\frac{C'ab}{lm}=\frac{B'c^2+C'b^2}{-l^2}=\frac{C'a^2+A'c^2}{-m^2}=\frac{A'b^2+B'a^2}{-n^2},$$

all of which are satisfied by $\frac{A'}{amn}=\frac{B'}{bnl}=\frac{C'}{clm}$ with the condition $al+bm+cn=0$.

Also the coordinates of A' are $(-a, b, c)$, and this will lie on S_3 if $bcla^2+\ldots=0$, or if $al+bm+cn=0$.

The tangent to S_3 at A' is $-lbcax+mcaby+nabcz=0$, or $-lx+my+nz=0$, or coincides with the straight line bc.

The preceding results lead to the following construction:—if ABC be a triangle inscribed in a conic U, and the tangents at A, B, C form a triangle A'B'C', then, if through A' be drawn any straight line A'bc meeting CA, AB in b, c, and a conic V be inscribed in the triangle ABC touching the sides in a, b, c, the conics U, V will be such that the polar recipro-

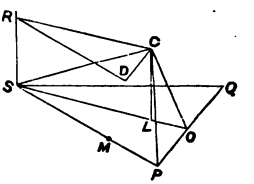
cal of either with respect to the other is the same conic W which passes through A'B'C' and touches at those points the straight lines A'*bc*, B'*ca*, C'*ab*. The three conics will form a harmonic system such that (1) the polar reciprocal of any one with respect to a second is the third, (2) any tangent to one is harmonically divided by the other two, (3) the tangents drawn from any point on one to the other two form a harmonic pencil. Also, if O be a common point of any two U, V, and OP, OQ tangents at O to U, V, then P, Q will be a common tangent to U, V, the points P, Q will lie on the third conic W, and OP, OQ will be the tangents to W at the points P, Q. This last property follows from the fact that the harmonic locus of two conics passes through the eight points of contact of their common tangents, and the harmonic envelope touches the eight tangents drawn to the two conics at their common points; and in this case the locus and envelope coincide. Any two such conics will have only two real common points and two real common tangents.

In conclusion, I may mention that all the above results may be proved geometrically in the case by which I first made acquaintance with harmonic systems, *i.e.*, a rectangular hyperbola, a circle whose centre is on the hyperbola and which passes through the centre of the hyperbola, and a parabola whose focus is the centre of the hyperbola and directrix the tangent to the hyperbola at the centre of the circle. We may then take the triangle ABC to be formed by the asymptotes and the directrix of the parabola, so that the circle is S_1, the hyperbola S_2, and the parabola S_3. The point *c* will be the centre of the circle, and the points *a*, *b* at ∞ on the asymptotes.

The general truth of the propositions then follows by projection.

4561. (By G. Sanderson.)—A great circle of a sphere is completely surrounded by 10 balls each 2 inches in diameter; and another row of 10 equal balls lies on the sphere, each one touching two of the first row and also touching each other; find the diameter of the second-row balls.

Solution by the Rev. J. R. Wilson, M.A.; Christine Ladd; *and others.*



Let S be the centre of the sphere; P, Q the centres of two of the 2-inch balls, which touch one another at the point O. Let C be the centre of the ball that touches the balls P, Q and rests upon the sphere; *r* its radius. Through C draw the plane CRD parallel to the plane SPQ, cutting SR perpendicular to the plane SPQ in the point R; and draw CL perpendicular to SO. Now PQ is the side of a regular polygon of 10 sides subtending an angle $\frac{2}{10}\pi$, or 36°, at S. It is evident, on a little consideration, that planes through SP, SQ perpendicular to the plane SPQ will touch the ball C at two points D, E in the plane RDC; and consequently CD subtends an angle of 18° at R. If M be the point of contact of the spheres S, P, we have

$$\text{radius of sphere S} = \text{SP} - \text{MP} = \operatorname{cosec} 18° - 1 = 5^{\frac{1}{2}}.$$

Also $$SC^2 - RC^2 = CL^2 = OC^2 - OL^2,$$

but $SC = 5^{\frac{1}{2}} + r$; $RC = r \operatorname{cosec} 18°$; $OC^2 = CP^2 - OP^2 = (1+r)^2 - 1$,

and $$OL = SO - SL = \cot 18° - r \operatorname{cosec} 18°.$$

Therefore, by substitution, we easily get

$$r = \frac{5^{\frac{1}{2}}(5^{\frac{1}{2}} - 1)}{(10 + 2.5^{\frac{1}{2}})^{\frac{1}{2}} - 2(5^{\frac{1}{2}} - 2)} = \cdot 829\ldots \text{ of an inch.}$$

[In the *Diary* for 1859 (Quest. 1943, pp. 73—77), solutions are given to a more general form of the problem, with a construction by the EDITOR for the latitudes of the several successive rows of spheres.]

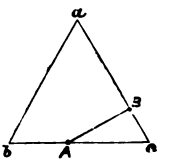
5439. (By ELIZABETH BLACKWOOD.)—Four random points A, B, C, D are taken in the perimeter of an equilateral triangle; show that the chance that the straight lines AB, CD will intersect *inside* the triangle (intersections on the perimeter *not* being counted within) is $\frac{8}{9}$.

Solution by C. LEUDESDORF, M.A.

Let A and B be first taken at random on the perimeter. They may fall (1) on the same side, (2) on adjacent sides; and the respective chances of these cases are $\frac{1}{3}$, $\frac{2}{3}$.

(1). When A, B fall on the same side, it is clear that no positions whatever of C, D will give a favourable case.

(2). When A, B fall on adjacent sides, as in the annexed Figure, then, for favourable cases, C and D must fall, one in AcB, and one in $AbaB$; yet not in such a way that they lie either both on ac, or both on bc, as in these cases we have intersection on the perimeter. The chance in this case then is



$$\frac{2}{9}\iint dx\,dy\,(AcB \,.\, AbaB - Ac \,.\, Ab - Bc \,.\, Ba)$$

$$= \frac{2}{9}\int_0^1\int_0^1 dx\,dy\,(x + y - xy)\ [\text{if } bc = 1,\ Ac = x,\ Bc = y] = \frac{1}{3}.$$

Thus the chance required $= \frac{1}{3} \,.\, 0 + \frac{2}{3} \,.\, \frac{1}{3} = \frac{2}{9}$.

5342. (By W. GALLATLY, M.A.)—If S be the focus of a conic inscribed in a triangle ABC, and any tangent meet the perpendiculars to SA, SB, SC in the points P, Q, R respectively, prove that AP, BQ, CR meet in a point.

Solution by CHRISTINE LADD.

Reciprocating the proposition, "If from any point on the circumference of a circle perpendiculars be drawn on the sides of any inscribed triangle, their three feet will lie in the same straight line," we have, "Lines drawn from the vertices of a circumscribing triangle to any tangent to a conic so as to subtend a right angle at the focus will meet in a point."

SYMBOLICAL LANGUAGE:—No. 2. *By* HUGH MCCOLL, B.A.

The method briefly sketched in my former article on this subject [see pp. 20—23 of this volume of the *Reprint*] is not restricted to Probability. It will also enable us to ascertain, by a sure, simple, and almost mechanical process, the new limits of integration which arise when we change the order of integration or the variables in a multiple integral. Take, for example, a multiple integral of 4 variables of the form

$$\int_{\lambda_7}^{\lambda_8} du \int_{\lambda_5}^{\lambda_6} dx \int_{\lambda_3}^{\lambda_4} dy \int_{\lambda_1}^{\lambda_2} dz \ \phi(u, x, y, z),$$

in which the limits λ_1, λ_2 are functions of the variables y, x, u; the limits λ_3, λ_4 functions of x, u; and so on. From this integral we get a compound statement of 8 factors (2 for each variable), namely, $p(u-\lambda_7)\, p'(u-\lambda_8)$ for the variable u, $p(x-\lambda_5)\, p'(x-\lambda_6)$ for the variable x, and so on. Suppose we have to *reverse* the order of integration: we must from these given statements deduce others in which the limits of u are functions of x, y, z, the limits of x functions of y, z, and so on. These deduced statements will constitute one or more terms of some such form as the following:

$$u_{1.2'.3'...0.4.7...}\, x_{6'.2'.3'...5.1.0}\, y_{0'.2'...3}\, z_{1'.0'.3},$$

in which u_0, x_0, $y_{0'}$, $z_{0'}$ are convenient equivalents for pu, px, $p'y$, $p'z$; while $u_{1'}$, $u_{2'}$, &c. are to be interpreted as in my former article.

We must now apply Rules 3, 4, 5 (see my former article) to the u-statements in every term. This will probably introduce fresh limits of x, y, z. We next apply Rules 3, 4, 5 to the x-statements in every term; this may introduce fresh limits of y, z; and so on. When we have thus applied Rules 3, 4, 5 successively to u, x, y, z, the required limits for the new order of integration will be obtained. The final result *may* consist of only one elementary term; but generally it will consist of more, and in complicated cases it may consist of several such terms. In Question 5373, for instance, the transformed integral required will be found to consist of *seven* terms, each with its own limits of integration, though the limits of u will be the same for all the terms.

Much labour may generally be saved by dropping *zero terms* (*i.e.*, terms with inconsistent factors) as we go along, when mere inspection of the table (without having recourse to Rule 5) will suffice to detect them. But if we overlook these zero terms, they will eventually disappear of themselves in the subsequent evolutions of the process. We may also shorten the process by cancelling factors which mere inspection of the table (instead of having recourse to Rule 3 or 4) will show to be implied in their co-factors *of the same variable.*

5352. (By W. C. Otter, F.R.A.S.) — A gentleman has a circular meadow whose area is just five acres, surrounded by an iron palisading, to the inside of which he wishes to tether his horse so as to enable him to graze over just *one* acre of ground: find the length of the tether.

Solution by S. Tebay, B.A.

First, let the radius of the circle $= 1$, r the length of the tether, and $2\theta\ (=2\alpha^\circ)$ the angle which the arc subtends at the centre of the circle. Then $r = 2\sin\frac{1}{2}\theta$, and the area pastured is readily found to be

$$\tfrac{1}{2}(\pi-\theta)r^2+\theta-\sin\theta = 2(\pi-\theta)\sin^2\tfrac{1}{2}\theta+\theta-\sin\theta = \frac{\pi}{n};\ \text{suppose;}$$

therefore $$\sin\theta+(\pi-\theta)\cos\theta = (1-n^{-1})\pi.$$

Let θ' be an approximate value of θ, and let

$$\sin\theta'+(\pi-\theta')\cos\theta'+e = (1-n^{-1})\pi,$$

e being the error. For θ write $\theta'+\delta\theta'$, and reduce; then we find

$$\delta\theta' = -\frac{e}{(\pi-\theta')\sin\theta'}.$$

Let $n = 5$; then $\sin\theta'+(\pi-\theta')\cos\theta'+e = 2{\cdot}51328$.

On trial it will be found that $\alpha = 40^\circ$, nearly, which makes $e = -00124$, $\delta\alpha = 2'42''$, and $\alpha = 40^\circ 2'42''$.

The radius of the circle and the length of the tether are, respectively,

$$\left(\frac{4840\times 5}{3{\cdot}1416}\right)^{\frac{1}{2}},\ \text{and}\ 2\sin\tfrac{1}{2}\alpha\left(\frac{4840\times 5}{3{\cdot}1416}\right)^{\frac{1}{2}} = 60{\cdot}0836 \text{ yards}.$$

3156. (By Professor Evans, M.A.)—Find a positive integral value of x that will satisfy the condition $94x^2+57x+34 = \square$.

Solution by Dr. Hart.

From $94x^2+57x+34$ subtract $256x^2$, and the remainder is

$$-162x^2+57x+34 = (54x+17)(-3x+2),$$

whence $$(16x)^2+(54x+17)(-3x+2) = \square = \left\{16x-\frac{m}{n}(-3x+2)\right\}^2.$$

Reducing, we find $$x = \frac{2m^2-17n^2}{3m^2-32mn+54n^2}.$$

Put the denominator $= 1$, then transposing and dividing by 3,

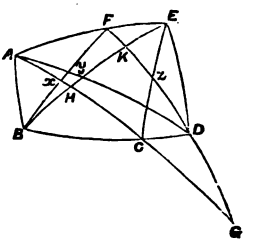
$$m^2-\frac{32}{3}mn = \frac{1-54n^2}{3},\quad \text{whence } m = \frac{16n\pm(94n^2+3)^{\frac{1}{2}}}{3}.$$

Let $n = 23$, then $m = 197$, and $x = 68625$. In all cases of such problems, by closely examining the coefficients and signs in the general equation,

we can set down at once the general value of x, and thus save much labour. In like manner, the general value of m can readily be set down. It will also be seen that there is no need of first finding a negative value of x, as has been done in the solutions given on pp. 39 and 40 of Vol. XXIV. of the *Reprint*, but that a positive value of x can be found directly.

5092. (By R. Battle, M.A.)—If a given spherical quadrilateral be divided by an arc of a great circle into two quadrilaterals, prove independently of the principles of gnomonical projections that the intersections of the diagonals of these two are on the circumference of a great circle which passes through a given point.

Solution by the Proposer.



Let ABDE be the given figure, CF any arc dividing it into two quadrilaterals ABCF and CDEF, x and z the intersections of the diagonals of these two respectively, and y the intersection of the diagonals of the given quadrilateral; also let x', y', z' be the other intersections of the three pairs of diagonals just named, and let AC and FD meet at G.

The points x', x', z', z, are manifestly on the circumference of a great circle, and by a well known theorem in Spherics, we have

$$\left.\begin{aligned} \sin CG \,.\, \sin HE \,.\, \sin Kz &= \sin Gz \,.\, \sin KE \,.\, \sin HC \\ \sin Gx \,.\, \sin HB \,.\, \sin KF &= \sin GF \,.\, \sin KB \,.\, \sin Hx \\ \sin GA \,.\, \sin Hy \,.\, \sin KD &= \sin GD \,.\, \sin Ky \,.\, \sin HA \end{aligned}\right\}.$$

Compounding these equations, we get

$$\sin GC \,.\, \sin GA \,.\, \sin Gx \,.\, \sin HE \,.\, \sin HB \,.\, \sin Hy \,.\, \sin KD \,.\, \sin KF \,.\, \sin KZ = \sin GF \,.\, \sin GD \,.\, \sin Gz \,.\, \sin KB \,.\, \sin KE \,.\, \sin Ky \,.\, \sin HA \,.\, \sin HC \,.\, \sin Hx \quad \ldots\ldots\ldots (1).$$

Again, by the same theorem, we have

$$\left.\begin{aligned} \sin GC \,.\, \sin HB \,.\, \sin KD &= \sin GD \,.\, \sin KB \,.\, \sin HC \\ \sin GA \,.\, \sin HE \,.\, \sin KF &= \sin GF \,.\, \sin KE \,.\, \sin HA \end{aligned}\right\};$$

whence $\sin GC \,.\, \sin GA \,.\, \sin HB \,.\, \sin HE \,.\, \sin KD \,.\, \sin KF = \sin GD \,.\, \sin GF \,.\, \sin KB \,.\, \sin KE \,.\, \sin HA \,.\, \sin HC \ldots (2).$

Now (2) reduces (1) to $\sin Gx \,.\, \sin Hy \,.\, \sin Kz = \sin Gz \,.\, \sin Ky \,.\, \sin Hx$,

and therefore the points x, y, z are on the circumference of a great circle, by the theorem above quoted; that is, the intersections x, x', z', z are on the circumference of a great circle that passes through the fixed point y, and consequently through its diametrically opposite point y'; which proves the theorem.

5340. (By C. LEUDESDORF, M.A.)—Find an expression for s_n, the sum of the n^{th} powers of the numbers less than a given number N, and prime to it; and hence prove that, when n is odd, the equation

$$(s_0\, s_1\, s_2 \ldots s_n \overline{\mathrm{Q}}\, x, 1)^n = 0$$

is satisfied by $x = -\frac{s_1}{s_0}$, and that the other roots are all of the form $-\frac{s_1}{s_0} \pm y$. [This includes, as a particular case, Question 5262, for which see Vol. XXVIII. of the *Reprint*, pp. 45—47.]

I. *Solution by* S. ROBERTS, M.A.; Prof. EVANS, M.A.; *and others.*

We may generalize the question still further by writing for "prime to it" the words "not divisible by any of p given factors of N." For if m_1 is a factor of N, the multiples of m_1 less than N are

$$m_1 \left(1, 2, 3, \ldots\ldots \frac{\mathrm{N}}{m_1} - 1\right),$$

and the sum of the n^{th} powers of numbers less than N, and not divisible by m_1, is

$$\Sigma\,(\mathrm{N}-1)^n - m_1^n\, \Sigma \left(\frac{\mathrm{N}}{m_1} - 1\right)^n.$$

If m_2 is another factor of N, the sum of the n^{th} powers of numbers less than N and not divisible by either m_1 or m_2 is in like manner

$$\Sigma\,(\mathrm{N}-1)^n - m_1^n\, \Sigma \left(\frac{\mathrm{N}}{m_1} - 1\right)^n - m_2^n\, \Sigma \left(\frac{\mathrm{N}}{m_2} - 1\right)^n + (m_1 m_2)^n\, \Sigma \left(\frac{\mathrm{N}}{(m_1 m_2)} - 1\right)^n,$$

where $(m_1 m_2)$ denotes the least common multiple of m_1, m_2. And generally, proceeding in this way, we get

$$s_n = \Sigma\,(\mathrm{N}-1)^n - \Sigma m_1^n\, \Sigma \left(\frac{\mathrm{N}}{m_1} - 1\right) + \Sigma\,(m_1 m_2)^n\, \Sigma \left(\frac{\mathrm{N}}{(m_1 m_2)} - 1\right)^n$$

$$- \Sigma\,(m_1 m_2 m_3)^n\, \Sigma \left(\frac{\mathrm{N}}{m_1 m_2 m_3} - 1\right)^n + \&\mathrm{c.} \pm (m_1 m_2 \ldots m_p)\, \Sigma \left(\frac{\mathrm{N}}{(m_1 m_2 \ldots m_p)} - 1\right)^n.$$

This is a linear function of terms of the form $\mathrm{M}^n\, \Sigma \left(\frac{\mathrm{N}}{\mathrm{M}} - 1\right)^n$, where M is a factor of N. We have next to consider a property of the sums of powers of the natural numbers up to P−1. Since, when n is odd,

$$(\tfrac{1}{2}\mathrm{P} - x)^n = (\tfrac{1}{2}\mathrm{P})^n - nx\,(\tfrac{1}{2}\mathrm{P})^{n-1} + \tfrac{1}{2}n\,(n-1)\,x^2\,(\tfrac{1}{2}\mathrm{P})^{n-2} - \ldots - x^n,$$

and $\Sigma_{x=1}^{x=\mathrm{P}-1}\,(\tfrac{1}{2}\mathrm{P} - x)^n = 0$, we have

$$(\mathrm{P}-1)\,(\tfrac{1}{2}\mathrm{P})^n - n\Sigma\,(\mathrm{P}-1)\,(\tfrac{1}{2}\mathrm{P})^{n-1} + \tfrac{1}{2}(n\,.\,n-1)\,\Sigma\,(\mathrm{P}-1)^2\,(\tfrac{1}{2}\mathrm{P})^{n-2} - \ldots - \Sigma\,(\mathrm{P}-1)^n = 0;$$

and, if Q is a factor of P, we have from this

$$\left(\frac{\mathrm{P}}{\mathrm{Q}} - 1\right)(\tfrac{1}{2}\mathrm{P})^n - n\mathrm{Q}\,\Sigma \left(\frac{\mathrm{P}}{\mathrm{Q}} - 1\right)(\tfrac{1}{2}\mathrm{P})^{n-1} + \tfrac{1}{2}n\,(n-1)\,\mathrm{Q}^2\,\Sigma \left(\frac{\mathrm{P}}{\mathrm{Q}} - 1\right)^2 (\tfrac{1}{2}\mathrm{P})^{n-2} - \ldots - \mathrm{Q}^n\,\Sigma \left(\frac{\mathrm{P}}{\mathrm{Q}} - 1\right)^n = 0 \ldots \text{(A)}.$$

Now if a series of equations have a common root, any linear function of these has the same root. It follows that any linear function of the expressions $(N-1)\,x^n + n\Sigma(N-1)\,x^{n-1} + \ldots + \Sigma(N-1)^n$,

$$\left(\frac{N}{m_1}-1\right)x^n + nm_1\Sigma\left(\frac{N}{m_1}-1\right)x^{n-1} + \ldots + n_1^n\Sigma\left(\frac{N}{m_1}-1\right)^n,$$

$$\left(\frac{N}{m_2}-1\right)x^n + nm_2\Sigma\left(\frac{N}{m_2}-1\right)x^{n-1} + \ldots + m_2^n\Sigma\left(\frac{N}{m_2}-1\right)^n,$$

&c. &c.

has $-\frac{1}{2}N$ for a root. Hence *a fortiori*

$$(s_0 s_1 s_2 \ldots s_n)(x+1)^n = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (a)$$

has $-\frac{1}{2}N$ for a root, and

$$-\frac{s_1}{s_0} = -\frac{\Sigma(N-1)}{N-1} = -\frac{m_1\Sigma\left(\frac{N}{m_1}-1\right)}{\frac{N}{m_1}-1} = \&c. = -\tfrac{1}{2}N.$$

Since n may be any odd number, it is plain that

$$(s_0 s_1 s_2 \ldots s_n)(y - \tfrac{1}{2}N,\ 1)^n = 0$$

contains only odd powers of y; or, y being divided out, the remaining equation contains only even powers of y. Hence the remaining roots of (a) are of the form $-\frac{s_1}{s_0} \pm y$. The basis of the result is (A).

II. *Solution by* J. Osborne, L. W. Jones, B.A., *and others.*

Let the prime factors of N be $a, b, c, \ldots$; and let $s_{np} = 1^n + 2^n \ldots + p^n$. Now numbers up to N divisible by a are $a, 2a, 3a, \ldots \frac{N}{a}a$; and therefore the sum of the n^{th} powers of these numbers $= a^n S_{nNa^{-1}}$. Similarly,

$a^n b^n S_{nNa^{-1}b^{-1}}$ = sum of n^{th} powers up to N divisible by ab;

$a_n b^n c^n S_{nNa^{-1}b^{-1}}$ = ,, ,, ,, abc.

But $s_n = S_{nN} - \Sigma a^n S_{nNa^{-1}} + \Sigma a^n b^n S_{nNa^{-1}b^{-1}} - \Sigma a^n b^n c^n S_{nNa^{-1}b^{-1}c^{-1}} + \ldots\ldots$ $\ldots\ldots(1)$,

Σ denoting the sum of certain quantities of which $a^n S_{nNa^{-1}}$, $a^n b^n S_{nNa^{-1}b^{-1}}$, &c. are types.

But $S_{nN} = N^n + \Sigma N^n$

$$= \frac{N^{n+1}}{n+1} + \tfrac{1}{2}N^n + \frac{B_1}{2!}nN^{n-1} - \frac{B_3}{4!}n(n-1)(n-2)N^{n-3} + \ldots\ldots;$$

also $a^n S_{nNa^{-1}}$

$$= \frac{N^{n+1}}{n+1}a^{-1} + \tfrac{1}{2}N^n a^0 + \frac{B_1}{2!}nN^{n-1}a - \frac{B_3}{4!}n(n-1)(n-2)N^{n-3}a^3 \ldots\ldots;$$

and therefore $\Sigma a^n S_{nNa^{-1}}$

$$= \frac{N^{n+1}}{n+1}\Sigma a^{-1} + \tfrac{1}{2}N^n \Sigma a^0 + \frac{B_1}{2!}nN^{n-1}\Sigma a - \frac{B_3}{4!}n(n-1)(n-2)N^{n-3}\Sigma a^3 + \ldots.$$

Similarly $\Sigma a^n b^n S_{nNa^{-1}b^{-1}}$

$$= \frac{N^{n+1}}{n+1}\Sigma a^{-1}b^{-1} + \tfrac{1}{2}N^n \Sigma a^0 b^0 + \frac{B_1}{2!} nN^{n-1}\Sigma ab - \frac{B_3}{4!} n(n-1)(n-2)N^{n-3}\Sigma a^3 b^3 \ldots\ldots,$$

and so on; therefore, by (1),

$$s_n = \frac{N^{n+1}}{n+1}\pi(1-a^{-1}) + \tfrac{1}{2}N^n \pi(1-a^0) + \frac{B_1}{2!} nN^{n-1}\pi(1-a) - \frac{B_3}{4!} n(n-1)(n-2)N^{n-3}\pi(1-a^3) + \ldots\ldots,$$

where $\pi(1-a^r) = (1-a^r)(1-b^r)\ldots = 1-\Sigma a^r + \Sigma a^r b^r - \Sigma a^r b^r c^r + \ldots,$

also $\pi(1-a^0) = \pi(1-1) = 0.$

If n is even, the last term is $(-1)^{\frac{1}{2}(n+2)} B_{n-1} N \pi(1-a^{n-1})$; and if n is odd, the last term is $(-1)^{\frac{1}{2}(n+1)} \frac{B_{n-2}}{2!} nN^2\pi(1-a^{n-2})$; therefore, if n is odd

$$s_n = \frac{N^{n+1}}{n+1}\pi(1-a^{-1}) + \frac{B_1}{2!} nN^{n-1}\pi(1-a) - \frac{B_3}{4!} n(n-1)(n-2)N^{n-3}\pi(1-a^3) + \ldots\ldots + (-1)^{\frac{1}{2}(n+1)} \frac{B_{n-2}}{2!} nN^2\pi(1-a^{n-2}).$$

In the expression $(s_0, s_1, s_2, \ldots s_n \overline{\times} x, 1)^n$, consider the coefficient of $B_{n-r}\pi(1-a^{n-r})$, n being odd, and therefore r even. It is

$$(-1)^{\frac{1}{2}(n-r+3)} \left\{ N^r n(n-1)\ldots(r+1) + nx \cdot N^{r-1}[(n-1)(n-2)\ldots r] + \frac{n(n-1)}{2!} x^2 N^{r-2}[(n-2)(n-3)\ldots(r-1)] + \ldots\ldots + [n(n-1)(n-2)\ldots r]x^{r-1}\cdot N \right\}$$

$$= (-1)^{\frac{1}{2}(n-r+3)} \{n(n-1)(n-2)\ldots(r+1)\} \{(N+x) - x^r\};$$

therefore $(s_0, s_1, s_2, \ldots s_n \overline{\times} x, 1)^n$

$$= \frac{\pi(1-a^{-1})}{n+1}[(N+x)^{n+1} - x^{n+1}] + \&c. \ldots\ldots + (-1)^{\frac{1}{2}(n+1)} \frac{B_{n-2}}{2!} n[(N+x)^2 - x^2]\pi(1-a^{n-2}) = 0.$$

Hence the equation is satisfied by $N+x = -x$, as the whole equation is divisible by $[(N+x)^2 - x^2]$; therefore it is satisfied by $x = -\frac{1}{2}N = -\frac{s_1}{s_0}$.

If, for x, $-\frac{1}{2}N + y$ is written, the equation becomes

$$\frac{\pi(1-a^{-1})}{n+1}[(\tfrac{1}{2}N+y)^{n+1} - (\tfrac{1}{2}N-y)^{n+1}] + \ldots\ldots + \ldots\ldots (-1)^{\frac{1}{2}(n+1)} \frac{B_{n-2}}{2!} n[(\tfrac{1}{2}N+y)^2 - (\tfrac{1}{2}N-y)^2]\pi(1-a^{n-2}) = 0,$$

which equation is also got if we change the sign of y; *i.e.*, the other roots are of the form $-\frac{s_1}{s_0} \pm y$.

LETTER, WITH PROBLEM: BY PROFESSOR SYLVESTER, F.R.S.

"Johns Hopkins University, Baltimore,
"October 11th, 1877.

"Dear Sir,—I should be glad if some of your readers would consider the following question, which has long perplexed me. It is almost *axiomatic* in the new theory of invariants, and yet I do not see my way to a proof of it. Let us agree to call the products

$$(1-2a\cos\theta+a^2)(1-2a\cos 3\theta+a^2)\ldots\ldots\{1-2a\cos(2i+1)\theta+a^2\},$$

$$(1-a)(1-2a\cos 2\theta+a^2)(1-2a\cos 4\theta+a^2)\ldots\ldots(1-2a\cos 2i\theta+a^2),$$

Cascades of degrees $2i+1$, $2i$ respectively, a being called the parameter and θ the argument. Now let us take any number of such cascades with the same argument, but each with its own parameter and having its own degree, which may be any given number whatever; and let us call $\Pi(\theta)$ the product of these cascades. Then it is obvious that $\frac{\sin\theta}{\Pi(\theta)}$ may be expanded in an infinite series according to positive integer powers of the *parameters*, and according to sines of multiples of the argument θ.

"It is an undoubted fact that this infinite series is omnipositive, that is, that every coefficient in it will be positive; and this is what I want to be proved. I need scarcely observe that if, instead of $\frac{\sin\theta}{\Pi(\theta)}$, we take $\frac{1}{\Pi(\theta)}$ and expand *it* according to powers of the parameters and *cosines* of multiples of the argument, this series will also be omnipositive, and that the proof is immediate and does not offer the slightest difficulty. It is the introduction of the numerator $\sin\theta$ which creates the difficulty. If this question can be solved, an important proposition in the theory of partitions will be demonstrated, which, for the case of a single *cascade*, may be put under the following form:—

"If n be any number not greater than $\frac{1}{2}ij$, and if we find first the number of partitions of n into i or fewer parts none greater than j, and secondly the number of partitions of $n-1$ into i or fewer parts none greater than j, the first number may be equal to, but can never be less than, the second.—Yours very truly,

"J. J. SYLVESTER."

[After the foregoing letter was in type, Professor SYLVESTER wrote to the EDITOR to state that he had discovered the proof in question, but that he still desired the letter to be published.]

5438. (By the EDITOR.)—If A, B, C, D be taken at random on the perimeter of a regular n-gon, find the respective probabilities that AB, CD will intersect (1) inside, (2) on, (3) outside the perimeter; also (4) what these values become when n is increased without limit; and (5) deduce therefrom solutions to Questions 5313, 5439, 5461.

Solution by C. LEUDESDORF, M.A.

Let A and B be first taken at random on the perimeter. If n be odd, they may fall on the same side, on sides next to one another, on sides next

but one to one another, ... on sides next but $\frac{1}{2}(n-1)-1$ to one another; and the respective chances of these cases will be $\frac{1}{n}, \frac{2}{n}, \frac{2}{n} \ldots \frac{2}{n}$.

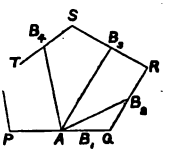
If n be even, they may fall on the same side, on sides next to one another' . . . . on sides next but $(\frac{1}{2}n-1)-1$ to one another, or on sides opposite to one another; and the respective chances of these cases will be

$$\frac{1}{n}, \frac{2}{n}, \frac{2}{n} \ldots \frac{2}{n}, \frac{1}{n}.$$

(1). Let A and B fall on the same side. Then intersections *inside* the perimeter are impossible. For intersections *on* the perimeter, C and D, may fall both on one of the sides adjacent to AB, or we may have either C or D falling on the side AB, and the other anywhere on the perimeter, except on the side AB, in which case the intersection is indeterminate case. Therefore, if p_1, p_2, p_3 denote the chances of an intersection within, on, without the perimeter, and if p_4 denote the chance of an indeterminate intersection, we have

$$p_4 = \frac{1}{n^2}, \quad p_1 = 0, \quad p_2 = \frac{2}{n^2} + \frac{2(n-1)}{n^2} = \frac{2}{n}, \quad \text{and } \therefore \; p_3 = 1 - \frac{2}{n} - \frac{1}{n^2}.$$

(2). Let A and B fall on sides next to one another, as PQ, QR in the annexed figure. Then for perimeter intersection, C and D must fall both in PQ or both in QR: thus in this case $p_2 = \frac{2}{n^2}$.



For internal intersection C may fall in AQB and D in the remaining part of the perimeter, or *vice versâ*; only they must not fall in such a way as to lie both on PQ, or both in QR.

If PQ = 1, QA = x, QB = y, we have

$$p_1 = \frac{2}{n^2}\int_0^1\int_0^1 \{(x+y)(n-x-y)-x(1-x)-y(1-y)\}\,dx\,dy = \frac{2n-3}{n^2}.$$

Now $p_4 = 0$, therefore $p_3 = 1 - \frac{2}{n^2} - \frac{2n-3}{n^2} = \left(1-\frac{1}{n}\right)^2$.

(3). Let A and B fall on sides next but one to one another; as PQ, RS in the figure. Then, for perimeter intersection, C and D must fall both in PQ or both in RS; thus as before $p_2 = \frac{2}{n^2}$. For internal intersection C and D must fall, one in AQRB, one in the remaining part of the perimeter; yet not in such a way as to fall both in PQ, or both in RS. Thus, if AQ = x, BR = y, we have

$$p_1 = \frac{2}{n^2}\int_0^1\int_0^1 \{(1+x+y)(n-1-x-y)-x(1-x)-y(1-y)\}\,dx\,dy = \frac{4n-9}{n^2}.$$

But $p_4 = 0$, therefore $p_3 = 1 - \frac{2}{n^2} - \frac{4n-9}{n^2} = 1 - \frac{4}{n} + \frac{7}{n^2}$.

(4). Let A and B fall on sides next but two to one another, as PQ, ST in figure. Then, just as in (2) and (3), we have $p_4 = 0$, $p_2 = \frac{2}{n^2}$,

$$p_1 = \frac{2}{n^2}\int_0^1\int_0^1 \{(2+x+y)(n-2-x-y)-x(1-x)-y(1-y)\}\,dx\,dy = \frac{6n-19}{n^2},$$

and

$$p_3 = 1 - \frac{2}{n^2} - \frac{6n-19}{n^2} = 1 - \frac{6}{n} + \frac{17}{n^2}.$$

And similarly, if A and B fall on sides next but 3, 4, &c., to one another. If then n be *odd*, the chance of internal intersection is

$$P_1 = 0 \cdot \frac{1}{n} + \left\{ \frac{2n-3}{n^2} \cdot \frac{2}{n} + \frac{4n-9}{n^2} \cdot \frac{2}{n} + \frac{6n-19}{n^2} \cdot \frac{2}{n} + \ldots \text{ to } \frac{n-1}{2} \text{ terms} \right\}$$

$$= \frac{2}{n^3}\left\{2n(1+2+3+\ldots) - 2(1^2+2^2+3^2+\ldots) - (1+1+1+\ldots)\right\} = \frac{n^3-4n+3}{3n^3}$$

If n be *even*, we have

$$P_1 = 0 \cdot \frac{1}{n} + \left\{ \frac{2n-3}{n^2} \cdot \frac{2}{n} + \frac{4n-9}{n^2} \cdot \frac{2}{n} + \ldots \text{ to } \frac{n}{2} - 1 \text{ terms} \right\}$$

$$+ \tfrac{1}{2} (\tfrac{1}{2}n\text{th term of foregoing series})$$

$$\frac{2}{n^3} = \left\{2n(1+2+\ldots) - 2(1^2+2^2+\ldots) - (1+1+\ldots)\right\} - \tfrac{1}{2}n^2 - 1 = \frac{n^3-4n+3}{3n^3}.$$

If n be *odd*, the chance of perimeter intersection is

$$P_2 = \frac{2}{n} \cdot \frac{1}{n} + \left\{ \frac{2}{n^2} \cdot \frac{2}{n} + \frac{2}{n^2} \cdot \frac{2}{n} + \ldots \text{ to } \tfrac{1}{2}(n-1) \text{ terms} \right\} = \frac{4n-2}{n^3};$$

and if n be *even*,

$$P_2 = \frac{2}{n} \cdot \frac{1}{n} + \left\{ \frac{2}{n^2} \cdot \frac{2}{n} + \frac{2}{n^2} \cdot \frac{2}{n} + \ldots \text{ to } \tfrac{1}{2}n - 1 \text{ terms} \right\} + \frac{2}{n^2} \cdot \frac{1}{n} = \frac{4n-2}{n^3}.$$

Evidently the chance of an indeterminate intersection, P_4, is $\frac{1}{n^3}$.

Therefore the chance of an external intersection is

$$1_3 = 1 - P_1 - P_2 - P_4 = \frac{2n^3-8n}{3n^3}.$$

Thus, whether n be odd or even, we have

$$P_1 = \frac{1}{3} - \frac{4}{3n^2} + \frac{1}{n^3}, \quad P_2 = \frac{4}{n^2} - \frac{2}{n^3}, \quad P_3 = \frac{2}{3} - \frac{8}{3n^2}, \quad P_4 = \frac{1}{n^3}.$$

5. When $n = 3$, these have the values $\frac{3}{9}$, $\frac{12}{27}$, $\frac{12}{27}$, $\frac{1}{27}$; which is the case of Question 5439. When $n = 4$, they have the values $\frac{17}{64}$, $\frac{7}{32}$, $\frac{1}{2}$, $\frac{1}{64}$; which is the case of Question 5313. When $n = \infty$, the polygon becomes a circle: in this case $P_1 = \frac{1}{24}$, $P_3 = \frac{23}{24}$, and the other chances are zero. This last result, which may be compared with Quest. 5461, may be readily verified: for suppose A to be taken as fixed on the circumference of the circle and $\angle AOB = \theta$. Then, for internal intersection, C and D must fall, one in the arc AB, one in the remaining part of the circumference;

therefore $P_1 = \frac{1}{2\pi}\int_0^{2\pi} \frac{2\theta(2\pi-\theta)}{4\pi^2} d\theta = \frac{1}{3}$; therefore $P_3 = \frac{2}{3}$,

P_2 and P_4 being of course zero.

5461. (By the Editor.) — All possible chords of a given circle being supposed equally probable; required the probabilities (1) of the intersection of two of them taken at random being within the circle, (2) of the intersections of three of them taken at random being all within the circle.

Solution by ELIZABETH BLACKWOOD.

1. First, take the case of 2 random chords AA′, BB′. All positions of the 4 points A, A′, B, B′ on the circumference are equally probable; but the chance will evidently be the same if we consider *one* of the points, as A, fixed. Then, always starting from A, and measuring in the same direction, the variable points A′, B, B′ give 2 favourable permutations BA′B′ and B′A′B, and the total number of possible permutations (all equally probable) is $1 \cdot 2 \cdot 3$; so that the chance is $\frac{2}{1 \cdot 2 \cdot 3} = \frac{1}{3}$.

2. Next, take 3 chords AA′, BB′, CC′. As before, consider A fixed. Let the chords AA′ and BB′ intersect, giving the permutation BA′B′. Let C fall between A and B; then the only favourable position of C′ is between A′ and B′, while there are 4 unfavourable positions of C′. This will be the case for any position of C when the permutation BA′B′ is given, and the same remark applies to the permutation B′A′B. Hence the chance of 3 intersections is $\frac{1}{3} \times \frac{1}{5} = \frac{1}{15}$.

Continuing the same manner of reasoning, the chance that n random chords will intersect within the circle is

$$\frac{1}{1 \cdot 3 \cdot 5 \cdot 7 \ldots (2n-1)}.$$

[Under a slightly different enunciation, this problem has been before discussed in various volumes of our *Reprints*, and with varying results. In Vol. V., p. 112 (Quest. 1894), and Vol. XXVI., p. 63 (Quest. 5059), the EDITOR and Mr. WOOLHOUSE obtain the value $\frac{1}{4}$ for the probability in (1), as found in the foregoing solution by Miss BLACKWOOD. Colonel CLARKE, however, in his solution of Quest. 2109 (Vol. XI., pp. 94, 95), by another view of random lines drawn across a closed convex curve, obtains for the probability in (1) the value $\frac{1}{3}$, and the same value was afterwards obtained by Professor CROFTON, and published in the *Philosophical Transactions* for 1868, p. 187, and in WILLIAMSON'S *Integral Calculus*, p. 331 (2nd ed.)

Again, for the case (2), Mr. ST. JOHN STEPHEN (Vol. XXVI., p. 48, Quest. 4975), and Professor TANNER (Vol. XXVI., p. 63, Quest. 5059), find the probability to be $\frac{1}{27}$; Professor CROFTON (*Phil. Trans.* for 1868, p. 197) finds the probability to be $\frac{4}{\pi^2} - \frac{1}{4}$, which is equal to $\frac{1}{7}$ nearly; and the EDITOR (Vol. II., p. 94, Quest. 1152), and Mr. WOOLHOUSE (Vol. V., p. 112), find the probability to be $\frac{1}{15}$, the same as Miss BLACKWOOD. On sending a proof of the foregoing Solution, and the Editorial Note thereon, to Mr. WOOLHOUSE, we received from him the following Note on the Question.]

NOTE ON QUESTION 5461. *By* W. S. B. WOOLHOUSE, F.R.A.S.

It is quite unnecessary to adduce any confirmation of Miss BLACKWOOD'S clear and satisfactory solution to Question 5461. In the solution to Question 4975 (*Reprint*, Vol. XXVI., p. 48), Mr. ST. JOHN STEPHEN correctly determines the probability that the intersection of two random chords

should lie within the circle $= \frac{1}{4}$; but this result does not properly lead to his supposed inference that the probability of three random chords mutually intersecting within the circle $= (\frac{1}{4})^3$. The separate chances of intersection not being independent should not be thus compounded by multiplication. To practically elucidate the nature of the error, Question 5502 is proposed as a new exercise. The second part of Prof. LLOYD TANNER'S solution to Question 5059 (*Reprint*, Vol. XXVI., p. 63) is not valid for an analogous reason.

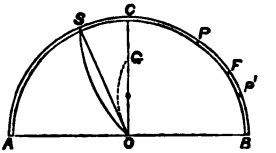
The general phases and corresponding probabilities of the various intersections and non-intersections of any stated number of random chords is fully and accurately investigated in my solution to Question 1894 (*Reprint*, Vol. V., p. 110).

Prof. CROFTON'S theory of random lines is peculiar and ingenious, and results derived from it should in general be regarded as special. See his statement (*Reprint*, Vol. VII., p. 84), and my remarks on random lines (*Reprint*, Vol. X., p. 33.)

5409. (By E. B. SEITZ.)—A semicircle is divided into two quadrants, and a point is taken at random in each quadrant; find the chance that their distance apart is less than the radius of the semicircle.

I. *Solution by* Colonel A. R. CLARKE, C.B., F.R.S.

Let the radius of the semicircle $= r$; and bisect the semicircle by the radius OC. Let the number of pairs of points—one in each quadrant—which are less than the distance r apart, be N. Increase r by dr, and let N receive an increment dN. Let $\angle COP = \theta$, (COF being $\frac{1}{3}\pi$). The element of the ring (between semicircles $r + dr$, r) at P is $r\,dr\,d\theta$. With centre P draw the arc OS; then $\angle SPO = \frac{1}{3}\pi$; area of segment $SO = \frac{1}{2}r^2(\frac{1}{3}\pi - \frac{1}{4}\sqrt{3})$; sector $SOC = \frac{1}{2}r^2(\frac{1}{3}\pi - \theta)$; therefore this curvilinear area $SOC = \frac{1}{2}r^2\{\frac{2}{3}\pi - \theta - \frac{1}{4}\sqrt{3}\}$. Now as P takes all positions from C to F, the number of pairs of points so added is



$$r\,dr\int_0^{\frac{1}{3}\pi} \tfrac{1}{2}r^2\{\tfrac{2}{3}\pi - \theta - \tfrac{1}{4}\sqrt{3}\}\,d\theta = \tfrac{1}{2}r^3\{(\tfrac{2}{3}\pi - \tfrac{1}{4}\sqrt{3})\tfrac{1}{3}\pi - \tfrac{1}{2}(\tfrac{1}{3}\pi)^2\}\,dr$$

$$= \tfrac{1}{2}r^3\{\tfrac{2}{9}\pi^2 - \tfrac{1}{18}\pi^2 - \tfrac{1}{4}\sqrt{3}\pi\}\,dr \ldots\ldots\ldots(1).$$

Again: for points such as P' between F and B let $BP' = \theta$. With centre P' draw the circular arc OG; then we have

area of segment $OG = r^2\theta - \frac{1}{2}r^2\sin 2\theta = r^2(\theta - \sin\theta\cos\theta)$,

and the sum of all the corresponding pairs is

$$r^3\,dr\int_0^{\frac{1}{3}\pi}(\theta - \sin\theta\cos\theta)\,d\theta = r^3\,dr\left[\tfrac{1}{2}\theta^2 - \tfrac{1}{2}\sin^2\theta\right]_0^{\frac{1}{3}\pi} = \tfrac{1}{2}r^3(\tfrac{1}{18}\pi^2 - \tfrac{1}{4})\,dr \ldots(2).$$

Adding this to the quantity (1), and doubling the sum so as to include the other quadrant AC of the ring, we get

$$dN = r^3\{\tfrac{7}{18}\pi^2 - \tfrac{1}{4}\sqrt{3}\pi - \tfrac{1}{4}\}\,dr, \quad N = \tfrac{1}{4}r^4\{\tfrac{7}{18}\pi^2 - \tfrac{1}{4}\sqrt{3}\pi^1 - \tfrac{1}{4}\}.$$

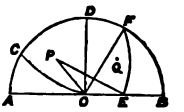
Also, total number of pairs $= (\frac{1}{4}\pi r^2)^2 = \frac{1}{16}\pi^2 r^4$.

Therefore the chance required is

$$\frac{16}{\pi^2 r^4} \cdot \frac{r^4}{4}\left\{\tfrac{7}{96}\pi^2 - \tfrac{1}{8}\sqrt{3}\pi - \tfrac{1}{4}\right\} = \frac{7}{9} - \frac{2}{\pi\sqrt{3}} - \frac{1}{\pi^2}.$$

II. *Solution by the* PROPOSER.

Let AOD and BOD be the two quadrants; P and Q the random points. With D, P as centres, and radii each equal to the radius of the semicircle, draw the arcs OC, EF. If P is in the surface OCD, and Q in the surface OEFD, their distance apart is less than the radius of the semicircle.



Let OA=1, OP=x, $\angle$POD=θ, and area OEFD$-u$; then we have

$$u = \tfrac{1}{2}\sin^{-1}(x\cos\theta) + \cos^{-1}(\tfrac{1}{2}x) - \tfrac{1}{4}\pi - \tfrac{1}{4}x(4-x^2)^{\frac{1}{2}} + \tfrac{1}{2}x\cos\theta(1-x^2\cos^2\theta)^{\frac{1}{2}} - \tfrac{1}{2}x^2\sin\theta\cos\theta.$$

But when P is in the surface OAC, the arc drawn from P as centre does not cut the arc BD, and the area cut off from the quadrant BOD is

$$u_1 = \tfrac{1}{2}\sin^{-1}(x\cos\theta) + \tfrac{1}{2}\cos^{-1}(x\sin\theta) + \tfrac{1}{2}x\cos\theta(1-x^2\cos^2\theta)^{\frac{1}{2}} - \tfrac{1}{2}x\sin\theta(1-x^2\sin^2\theta)^{\frac{1}{2}} - x^2\sin\theta\cos\theta.$$

Hence, since the whole number of ways the two points can be taken is $\frac{1}{16}\pi^2$, we have for the required chance

$$p = \frac{16}{\pi^2}\int_0^{\frac{1}{3}\pi}\int_0^1 u\,d\theta\,x\,dx + \frac{16}{\pi^2}\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi}\left\{\int_0^{2\cos\theta} ux\,dx + \int_{2\cos\theta}^1 u_1 x\,dx\right\}d\theta$$

$$= \frac{1}{2\pi^2}\int_0^{\frac{1}{3}\pi}(8\pi - 8\theta - \pi\sec^2\theta + 2\theta\sec^2\theta + 2\tan\theta - 6\sqrt{3})\,d\theta$$

$$+ \frac{1}{2\pi^2}\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi}(16\pi - 32\theta - \pi\sec^2\theta - 2\pi\operatorname{cosec}^2\theta + 2\theta\sec^2\theta + 2\tan\theta + 6\theta\operatorname{cosec}^2\theta - 6\cot\theta - 16\sin\theta\cos\theta)\,d\theta$$

$$= \frac{4}{3} - \frac{\sqrt{3}}{\pi} - \frac{2}{\pi^2}.$$

[Mr. SEITZ considers the first solution of this question to be "undoubtedly wrong," inasmuch as Colonel CLARKE "supposes the first point to take all possible positions in one of the quadrants, and, for each position, limits the second point to the quadrant whose radius is the distance of the first point from the centre, and finds the number of positions it may take, in each of which the distance between the points is less than the distance of the first point from the centre; so that, instead of solving the question proposed, he has in reality solved the following problem:—A semicircle is divided into two quadrants, and a point is taken at random in each quadrant; find the chance that their distance apart is less than the distance of the centre of the semicircle from the point nearest thereto. We may add here that Mr. SEITZ's result is, in decimals, ·5794, and Colonel CLARKE's ·3089.]

5482. (By S. Constable.)—Let ABC be a triangle (of area Δ) circumscribed by a circle, and let the arcs BC, CA, AB be bisected at A', B', C', forming a new triangle A'B'C' ($=\Delta_1$), and a hexagon AB'CA'BC' ($=\Delta_2$); also let Δ_3 denote the area of the triangle $O_1O_2O_3$, where O_1, O_2, O_3 are the escribed centres; then prove that

$$\Delta : \Delta_1 : \Delta_2 : \Delta_3 = r : \tfrac{1}{2}R : R : 2R.$$

Solution by Dr. Booth, F.R.S.; E. W. Symons; *and many others.*

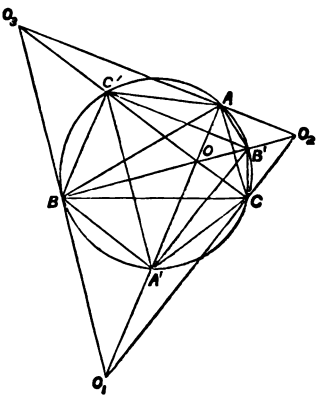
With the usual notation, we have

$$\Delta = rs,$$

$$\Delta_1 = \tfrac{1}{2}R^2(\sin A + \sin B + \sin C)$$

$$= \frac{R^2}{2} \cdot \frac{s}{R} = \tfrac{1}{2}Rs,$$

(see Booth's *New Geometrical Methods*, Vol. II., Sec. 185). We have also, for the hexagon AB'CA'BC', $\Delta_2 = Rs$.



Since the given triangle ABC is the orthocentric triangle of the principal excentral triangle $O_1O_2O_3$, the radius of the circle that circumscribes the triangle $O_1O_2O_3$ is 2R; and the area of any triangle is equal to the radius of the circumscribed circle multiplied by the semiperimeter of its orthocentric triangle (see the above cited volume, Secs. 194, 195, 216); hence $\Delta_3 = 2Rs$.

It is manifest therefore that these areas have to one another the given ratios.

5359. (By Professor Townsend, F.R.S.)—Two ordinary tangents being supposed drawn to a cuspidal cubic from any point on the cuspidal tangent, or to a tricuspidal quartic from any point on one of the three cuspidal tangents; shew that their chord of contact passes through the point of inflexion in the case of the cubic, and through the pole of the cuspidal tangent with respect to the bitangent chord in the case of the quartic.

Solution by the Proposer.

By reciprocation, from an arbitrary centre, the above, in both its parts, becomes transformed into the familiar property that, in a nodal cubic, the harmonic polar of the point of inflexion with respect to the curve coincides with the ordinary polar of the point with respect to the angle, real, evanescent, or imaginary, determined by the two nodal tangents; therefore, &c.

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